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ON THE USUAL PRODUCT OF RATIONAL ARITHMETIC FUNCTIONS

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The Dirichlet convolution of two arithmetic functions f and g is defined by

$$(f*g)(n) = \sum_{d|n} f(d)\,g(n/d) \qquad (n \in \mathbf{N}).$$

An arithmetic function f is called ([1], [6], [9]) *rational of degree* (n, m), where n and m are nonnegative integers, if

$$f = f_1 * \ldots * f_n * g_1^{-1} * \ldots * g_m^{-1}$$

for some completely multiplicative functions $f_1, \ldots, f_n, g_1, \ldots, g_m$. For nonnegative integers n, m let $R_{(n,m)}$ denote the set of all rational arithmetic functions of degree (n, m). In this note we shall prove the following

THEOREM. *If* $f \in R_{(n,m)}$ *and* $g \in R_{(t,r)}$, *then* $fg \in R_{(nt,M)}$, *where*

$$M = \begin{cases} nt - \max(n-m, t-r) & \text{if } 0 \leqslant m < n \text{ and } 0 \leqslant r < t, \\ nt + \max(m-n, r-t) & \text{if } m \geqslant n \text{ or } r \geqslant t \ (n, t \neq 0), \\ m & \text{if } n = 0,\ t > 0, \\ r & \text{if } t = 0,\ n > 0, \\ \min(m, r) & \text{if } n = t = 0. \end{cases}$$

The estimation cannot be improved.

Some partial results in this direction are contained in [3], [7], [9] (see also [4], [5]).

For every prime number p the generating series $f_p(z)$ of a multiplicative function f to the base p is defined by

$$f_p(z) = \sum_{n=0}^{\infty} f(p^n)\,z^n.$$

Each multiplicative function f is completely determined by its generating series. It is known (see [6], p. 45) that a multiplicative function f is rational of degree (n,m) if and only if for each prime p there exist complex numbers $r_1^{(p)},\ldots,r_m^{(p)},s_1^{(p)},\ldots,s_n^{(p)}$ such that

$$f_p(z)=\frac{1+r_1^{(p)}z+\ldots+r_m^{(p)}z^m}{1+s_1^{(p)}z+\ldots+s_n^{(p)}z^n}. \tag{1}$$

We shall investigate the generating series $(fg)_p(z)$ of the product fg of arithmetic functions f and g using the following theorem due to Hadamard: If

$$A(z)=\sum a_n z^n,\quad B(z)=\sum b_n z^n \quad \text{and} \quad C(z)=\sum a_n b_n z^n,$$

then

$$C(z)=\frac{1}{2\pi i}\int_\gamma A(s)B\left(\frac{z}{s}\right)\frac{ds}{s}, \tag{2}$$

where γ is a contour in the s plane which includes the singularities of $B(z/s)/s$ and excludes the singularities of $A(s)$ (see [2], p. 813, or [8], pp. 157–159).

LEMMA 1. *Let*

$$f_p(z)=\prod_{k=1}^{m}(1-a_k z)\prod_{l=1}^{n}(1-b_l z)^{-1},\quad g_p(z)=\prod_{i=1}^{r}(1-c_i z)\prod_{j=1}^{t}(1-d_j z)^{-1}, \tag{3}$$

where $0\leqslant m<n$, $0\leqslant r<t$, *and let* $b_l\neq 0$ $(l=1,\ldots,n)$, $b_{l_1}\neq b_{l_2}$ *for* $l_1\neq l_2$, $d_j\neq 0$ $(j=1,\ldots,t)$ *and* $d_{j_1}\neq d_{j_2}$ *for* $j_1\neq j_2$. *Then*

$$(fg)_p(z)=\frac{1+\sum_{\nu=1}^{M}\lambda_\nu z^\nu}{\prod_{l=1}^{n}\prod_{j=1}^{t}(1-b_l d_j z)} \tag{4}$$

for some complex numbers $\lambda_1,\ldots,\lambda_M$, *where* $M=nt-\max(n-m,t-r)$. *Moreover*, $\lambda_M\neq 0$ *for suitable* a_k, b_l, c_i, d_j.

Proof. Suppose $n-m\geqslant t-r$. The case $t-r\geqslant n-m$ is similar. Now, using Hadamard's theorem to the series $f_p(z)$ and $g_p(z)$ and the Cauchy residue theorem we get

$$(fg)_p(z)=\frac{1}{2\pi i}\int_\gamma f_p(s)g_p\left(\frac{z}{s}\right)\frac{ds}{s}$$

$$=\frac{1}{2\pi i}\int_\gamma\frac{\prod_{k=1}^{m}(1-a_k s)\prod_{i=1}^{r}(s-c_i z)}{\prod_{l=1}^{n}(1-b_l s)\prod_{j=1}^{t}(s-d_j z)}s^{t-r-1}\,ds$$

$$= \sum_{h=1}^{t} \frac{\prod_{k=1}^{m}(1-a_k d_h z)\prod_{i=1}^{r}(d_h-c_i)}{\prod_{l=1}^{n}(1-b_l d_h z)\prod_{\substack{j=1\\ j\neq h}}^{t}(d_h-d_j)} d_h^{t-r-1}$$

$$= \frac{1}{\prod_{h=1}^{t}\prod_{l=1}^{n}(1-b_l d_h z)} \sum_{h=1}^{t} \frac{\prod_{i=1}^{r}(d_h-c_i)}{\prod_{\substack{j=1\\ j\neq h}}^{t}(d_h-d_j)} d_h^{t-r-1} \prod_{k=1}^{m}(1-a_k d_h z) \prod_{\substack{u=1\\ u\neq h}}^{t}\prod_{l=1}^{n}(1-b_l d_u z).$$

This proves (4). In order to prove that $\lambda_M \neq 0$ for suitable a_k, b_l, c_i, d_j we consider the coefficient of the highest power of z in the numerator of the above fraction. It is equal to

$$\lambda_M = \sum_{h=1}^{t} \frac{\prod_{i=1}^{r}(d_h-c_i)}{\prod_{\substack{j=1\\ j\neq h}}^{t}(d_h-d_j)} d_h^{t-r-1} \prod_{k=1}^{m}(-a_k d_h) \prod_{\substack{u=1\\ u\neq h}}^{t}\prod_{l=1}^{n}(-b_l d_u)$$

$$= (-1)^{m+(t-1)n}\Big(\prod_{k=1}^{m} a_k\Big)\Big(\prod_{l=1}^{n} b_l\Big)^{t-1}\Big(\prod_{u=1}^{t} d_u\Big)^{n} \sum_{h=1}^{t} \frac{\prod_{i=1}^{r}(d_h-c_i)}{\prod_{\substack{j=1\\ j\neq h}}^{t}(d_h-d_j)} d_h^{t-r-1-n+m}.$$

Note that

$$g_p(z) = \sum_{h=1}^{t} \frac{A_h}{1-d_h z}, \qquad \text{where } A_h = d_h^{t-r-1} \frac{\prod_{i=1}^{r}(d_h-c_i)}{\prod_{\substack{j=1\\ j\neq h}}^{t}(d_h-d_j)}.$$

Therefore

$$\lambda_M = (-1)^{m+(t-1)n}\Big(\prod_{k=1}^{m} a_k\Big)\Big(\prod_{l=1}^{n} b_l\Big)^{t-1}\Big(\prod_{u=1}^{t} d_u\Big)^{n} \sum_{h=1}^{t} A_h d_h^{m-n}.$$

We have

$$\sum_{h=1}^{t} \frac{A_h}{1-d_h z} = \frac{\prod_{i=1}^{r}(1-c_i z)}{\prod_{j=1}^{t}(1-d_j z)}.$$

Substituting $-1/z$ for z in the above equality gives

$$\sum_{h=1}^{t} \frac{A_h}{z+d_h} = z^{t-r-1} \frac{\prod_{i=1}^{r} (z+c_i)}{\prod_{j=1}^{t} (z+d_j)}.$$

Hence

$$\sum_{h=1}^{t} A_h \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{d_h^{k+1}} = z^{t-r-1} \prod_{i=1}^{r} (z+c_i) \prod_{j=1}^{t} \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{d_j^{k+1}}.$$

Comparing the coefficients of z^{n-m-1} we see that $\sum_{h=1}^{t} A_h d_h^{m-n}$ need not vanish (it suffices to assume that $c_i > 0$ for $i = 1, \ldots, r$ and $d_j < 0$ for $j = 1, \ldots, t$). This proves that $\lambda_M \neq 0$ for suitable a_k, b_l, c_i, d_j and completes the proof of Lemma 1.

Remark 1. Note that the fraction on the right side of (4) can be irreducible. This follows from the observation that the number $z_0 = 1/(b_\beta d_\gamma)$, where $\beta \in \{1, \ldots, n\}$ and $\gamma \in \{1, \ldots, t\}$, is a root of the polynomial in the numerator of the fraction if and only if

$$\prod_{i=1}^{r} (d_\gamma - c_i) \prod_{k=1}^{m} \left(1 - \frac{a_k}{b_\beta}\right) \prod_{\substack{u=1 \\ u \neq \gamma}}^{t} \prod_{l=1}^{n} \left(1 - \frac{b_l d_u}{b_\beta d_\gamma}\right) = 0,$$

i.e.

$$\prod_{i=1}^{r} (d_\gamma - c_i) \prod_{k=1}^{m} (b_\beta - a_k) \prod_{\substack{u=1 \\ u \neq \gamma}}^{t} \prod_{l=1}^{n} (b_\beta d_\gamma - b_l d_u) = 0.$$

LEMMA 2. *Let $f_p(z)$ and $g_p(z)$ be given by (3), where $m \geqslant n$ or $r \geqslant t$ $(n, t \neq 0)$, and let $b_l \neq 0$ $(l = 1, \ldots, n)$, $b_{l_1} \neq b_{l_2}$ for $l_1 \neq l_2$, $d_j \neq 0$ $(j = 1, \ldots, t)$ and $d_{j_1} \neq d_{j_2}$ for $j_1 \neq j_2$. Then*

$$(5) \qquad (fg)_p(z) = \frac{1 + \sum_{\nu=1}^{M} \lambda_\nu z^\nu}{\prod_{l=1}^{n} \prod_{j=1}^{t} (1 - b_l d_j z)}$$

for some complex numbers $\lambda_1, \ldots, \lambda_M$, where $M = nt + \max(m-n, r-t)$. Moreover, $\lambda_M \neq 0$ for suitable a_k, b_l, c_i, d_j.

Proof. The series $f_p(z)$ and $g_p(z)$ can be written in the form

$$f_p(z) = \sum_{k=0}^{m-n} u_k z^k + \sum_{i=1}^{n} \frac{A_i}{1 - b_i z}, \qquad g_p(z) = \sum_{l=0}^{r-t} v_l z^l + \sum_{j=1}^{t} \frac{B_j}{1 - d_j z}.$$

Thus applying Hadamard's theorem we get

$$(fg)_p(z) = \frac{1}{2\pi i}\int_\gamma f_p(s)\, g_p\left(\frac{z}{s}\right)\frac{ds}{s}$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{m-n}\sum_{l=0}^{r-t} u_k v_l s^k \left(\frac{z}{s}\right)^l \frac{ds}{s} + \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{m-n}\sum_{j=1}^{t} u_k B_j \frac{s^k}{s-d_j z} ds$$

$$+ \frac{1}{2\pi i}\int_\gamma \sum_{l=0}^{r-t}\sum_{i=1}^{n} v_l A_i \frac{z^l}{(1-b_i s)\, s^{l+1}} ds + \frac{1}{2\pi i}\int_\gamma \sum_{i=1}^{n}\sum_{j=1}^{t} \frac{A_i B_j}{(1-b_i s)(s-d_j z)} ds.$$

We change the order of integration and summation and then evaluate the obtained integrals using the Cauchy residue theorem to get

$$(fg)_p(z) = \sum_{k=0}^{\min(m-n,r-t)} u_k v_k z^k + \sum_{k=0}^{m-n}\sum_{j=1}^{t} u_k B_j d_j^k z^k$$

$$+ \sum_{l=0}^{r-t}\sum_{i=1}^{n} v_l A_i b_i^l z^l + \sum_{i=1}^{n}\sum_{j=1}^{t} \frac{A_i B_j}{1-b_i d_j z}.$$

This proves (5). Moreover, after direct calculations we find that $\lambda_M \neq 0$ for suitable a_k, b_l, c_i, d_j (cf. the proof of Lemma 1). The proof of Lemma 2 is complete.

Remark 2. Note that the fraction on the right side of (5) can be irreducible. This can be verified as in Remark 1.

Proof of the Theorem. Let $f \in R_{(n,m)}$ and $g \in R_{(t,r)}$. Then the series $f_p(z)$ and $g_p(z)$ are given by (3). Relating to n, m, r, t we distinguish 3 cases.

Case 1. Suppose $0 \leqslant m < n$, $0 \leqslant r < t$. If $b_l \neq 0$ $(l = 1, \ldots, n)$, $b_{l_1} \neq b_{l_2}$ for $l_1 \neq l_2$, $d_j \neq 0$ $(j = 1, \ldots, t)$ and $d_{j_1} \neq d_{j_2}$ for $j_1 \neq j_2$, then by Lemma 1 the series $(fg)_p(z)$ is given by (4). Now, note that the coefficients of z^α $(\alpha = 0, 1, 2, \ldots)$ in the series $(fg)_p(z)$ and in the polynomial $\prod\prod(1-b_l d_j z)$ are polynomials in the variables a_k, b_l, c_i, d_j. Hence the coefficients λ_ν in (4) also are polynomials in the same variables. Therefore we may take the limits of both sides of (4) when $b_l \to 0$ $(l = 1, \ldots, n)$, $d_j \to 0$ $(j = 1, \ldots, t)$, $b_{l_1} \to b_{l_2}$ or $d_{j_1} \to d_{j_2}$. These operations do not raise the degrees of the polynomials $1+\sum\lambda_\nu z^\nu$ and $\prod\prod(1-b_l d_j z)$. This ends the proof of the Theorem in Case 1.

Case 2. Suppose $m \geqslant n$ or $r \geqslant t$ $(n, t \neq 0)$. Then applying Lemma 2 we can proceed as in Case 1 to arrive at the desired result.

Case 3. Suppose $n = 0$ or $t = 0$. If $n = 0$ and $t > 0$, then

$$(fg)_p(z) = \sum_{k=0}^{m} f(p^k)\, g(p^k)\, z^k$$

and, consequently, $fg \in R_{(0,m)}$. Also the coefficient of z^m is nonzero for suitable f and g [for example, take $f = \lambda^{-1} * \ldots * \lambda^{-1}$ (m factors), $g = N * \ldots$

$\ldots * N * \lambda^{-1} * \ldots * \lambda^{-1}$ (N, t times; λ^{-1}, r times), where λ is the Liouville function and $N(n) = n$ for all $n \in N$]. This proves the theorem in the case $n = 0$, $t > 0$. The proofs in the cases $n = t = 0$ and $n > 0$, $t = 0$ are similar. The proof of the Theorem is complete.

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