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NORTH-HOLLAND

## On Smith's Determinant

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### ABSTRACT

We give a brief review of papers relating to Smith's determinant and point out a common structure that can be found in many extensions and analogues of Smith's determinant. We present the common structure in the language of posets. We also make an investigation on a conjecture of Beslin and Ligh on greatest common divisor (GCD) matrices in the sense of meet matrices and give characterizations of the posets satisfying the conjecture. Further, we give a counterexample for the conjecture of Bourque and Ligh that the least common multiple matrix on any GCD-closed set is invertible. © Elsevier Science Inc., 1997

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## 1. INTRODUCTION

The classical Smith's [36] determinant evaluation is

$$\det [(i, j)]_{n \times n} = \phi(1)\phi(2) \cdots \phi(n), \quad (1.1)$$

where  $(i, j)$  is the greatest common divisor (GCD) of  $i$  and  $j$ , and  $\phi$  is Euler's totient function. Smith also evaluated more general determinants. In fact, let  $m_1, m_2, \dots, m_n$  be distinct positive integers such that  $d \mid m_i$  implies  $d = m_j$  for some  $j = 1, 2, \dots, n$ . Then

$$\det [f(m_i, m_j)]_{n \times n} = g(m_1)g(m_2) \cdots g(m_n), \quad (1.2)$$

where

$$f(m_i, m_j) = \sum_{d \mid (m_i, m_j)} g(d). \quad (1.3)$$

If, in particular,  $m_i = i$  for all  $i = 1, 2, \dots, n$ , then

$$\det [f(i, j)]_{n \times n} = g(1)g(2) \cdots g(n). \quad (1.4)$$

For  $g = \phi$ ,

$$f(i, j) = \sum_{d \mid (i, j)} \phi(d) = (i, j)$$

by the well-known property [34, p. 83]

$$\sum_{d \mid n} \phi(d) = n,$$

and thus (1.4) reduces to (1.1).

Since Smith's paper [36] of 1876 this field has been studied extensively. It seems that some modern authors are not thoroughly familiar with the results. In this paper we give an extensive list and a brief review of papers relating to Smith's determinant (see Section 2), and point out a common structure that can be found in many extensions and analogues of Smith's determinant (see Section 3). The common structure is presented in the language of posets. We also present meet-matrix analogues of certain results of Beslin and Ligh [6, 7]

and Li [23] on GCD matrices and make an investigation of a conjecture of Beslin and Ligh [7] (see Section 4). In Section 5, we give characterizations of the so-called *regular* posets, which are the posets satisfying the conjecture. In Section 6, we generalize the concept of regular posets. This generalization makes it possible to give a characterization of those posets  $S$  of  $n$  elements for which every  $n \times n$  submatrix of the incidence matrix  $E(S, \bar{S})$  is invertible [for definition of  $E(S, \bar{S})$ , see Section 4]. Further, in Section 7 we give a counterexample for the conjecture of Bourque and Ligh [8] that the least common multiple LCM matrix on any GCD-closed set is invertible.

For number-theoretic background and for previous general accounts of Smith's determinant, we refer to the books by McCarthy [28], Shapiro [33], and Sivaramakrishnan [34]. For the theory of posets, we refer to the books by Aigner [1] and Stanley [38].

## 2. ON PAPERS RELATING TO SMITH'S DETERMINANT

In this section we briefly review papers relating to Smith's determinant. Dickson [15, pp. 122–129] reports on several papers devoted to proofs and extensions of Smith's determinant. We do not consider these papers here.

A simple and elegant proof was suggested by Pólya and Szegő [31], who observed that  $[f(i, j)]_{n \times n}$  in (1.4) can be written in the form

$$[f(i, j)]_{n \times n} = BC^T, \quad (2.1)$$

where  $B$  and  $C$  are lower triangular matrices given by  $b_{ij} = g(j)$  if  $j \mid i$ , and  $= 0$  otherwise; and  $c_{ij} = 1$  if  $j \mid i$ , and  $= 0$  otherwise. Carlitz [11] gave some new insight into the structure of  $[f(i, j)]_{n \times n}$  in (1.4). For example, he observed [11, (17)] that

$$[f(i, j)]_{n \times n} = C \operatorname{diag}(g(1), g(2), \dots, g(n)) C^T, \quad (2.2)$$

where  $C$  is the triangular matrix given in (2.1). Gyires [17] observed (1.4) in the case  $f(i, j) = (i, j)^r$ , and Maurer and Veégh [27] proved this evaluation by induction on  $n$ . Castaldo [12] studied properties of the sequence  $\phi(1), \phi(2) \cdots \phi(n)$ ,  $n = 1, 2, \dots$ .

Jager [20, Theorem 5] introduced a unitary analogue of Smith's determinant. In fact, a divisor  $d$  of  $n$  with  $(d, n/d) = 1$  is said to be a unitary divisor

of  $n$  and is denoted by  $d \parallel n$ . Let  $(i, j)^*$  denote the greatest common unitary divisor of  $i$  and  $j$ . Jager observed that if

$$f(i, j) = \sum_{d \parallel (i, j)^*} g(d), \quad (2.3)$$

then

$$\det [f(i, j)]_{n \times n} = g(1)g(2) \cdots g(n). \quad (2.4)$$

Jager's proof is based on the observation that  $[f(i, j)]_{n \times n}$  can be written as a product of two triangular matrices. Let  $\phi^*(n)$  be the number of positive integers less than or equal to  $n$  that are not divisible by any of its unitary divisors ( $> 1$ ). By the formula

$$\sum_{d \parallel n} \phi^*(d) = n,$$

Jager obtained the evaluation

$$\det [(i, j)^*]_{n \times n} = \phi^*(1)\phi^*(2) \cdots \phi^*(n). \quad (2.5)$$

Nageswara Rao [30] gave an  $A$ -analogue of Smith's determinant, where  $A$  denotes Narkiewicz's regular convolution [28, Chapter 4]. Smith's determinant and its unitary analogue are special cases of Nageswara Rao's determinant. Generalizations in this direction have also been developed by Davison [14, pp. 43–44] and Wall [40].

Apostol [2, Theorem 9] observed that Smith's determinant also has connections with Ramanujan's sum and its generalizations. In fact, if  $g$  and  $h$  are arithmetical functions and

$$f(i, j) = \sum_{d \mid (i, j)} g(d)h(j/d), \quad (2.6)$$

then

$$\det [f(i, j)]_{n \times n} = g(1)g(2) \cdots g(n)h(1)^n. \quad (2.7)$$

Apostol also used the idea that  $[f(i, j)]_{n \times n}$  can be written as a product of two triangular matrices. In particular, taking  $g(n) = n$  for all  $n$  and  $h = \mu$ , where  $\mu$  is the Möbius function, Apostol obtained the evaluation

$$\det [c(i, j)]_{n \times n} = n!, \quad (2.8)$$

where  $c(i, j)$  is Ramanujan's sum. P. Kesava Menon [21, (5.7)] evaluated the determinant relating to Ramanujan's sum on the set of the divisors of a positive integer.

McCarthy [29] extended Apostol's evaluation for the so-called even arithmetical functions. Ramanujan's sum and its generalization given in (2.6) are even functions.

Daniloff's [13] analogue of Smith's determinant can be presented as follows. Let  $\Omega_k(n) = m$  if  $n = m^k$  for some positive integer  $m$ , and  $= 0$  otherwise. Let

$$f(i, j) = \sum_{d|(i, j)} \Omega_k(i/d)\Omega_k(j/d)g(d).$$

Then

$$\det [f(i, j)]_{n \times n} = g(1)g(2) \cdots g(n). \tag{2.9}$$

Poset-theoretic generalizations of Smith's determinant have been developed by Lindström [26], Rajarama Bhat [32], D. A. Smith [35], and Wilf [41]. In this paper we also consider matrices on posets as mentioned in the introduction.

Multidimensional Smith's determinants have been considered by Gegenbauer [16], Haukkanen [18, 19], Lehmer [22], Vaidyanathaswamy [39], and Sokolov [37]. We do not consider these papers here. For multidimensional determinants reference is made to the recent paper by Haukkanen [19].

Motivation for the above brief survey of old papers arises from the observation that some authors have recently begun to study this field intensively. This new inspiration may be considered to start from the papers by Beslin and Ligh [5, 6]. For other recent contributions, we refer to the papers by Ligh [25], Beslin and el-Kassar [4], Li [23, 24], Beslin [3], Beslin and Ligh [7], and Bourque and Ligh [8–10]. These papers contain, among other things, several structure theorems and determinant evaluations of GCD matrices. Many of these papers also contain conjectures and unsolved problems. In Sections 4, 5, and 6 of this paper we consider a conjecture of Beslin and Ligh [7].

It is less known that H. J. S. Smith [36, Section 3] also evaluated the determinant  $\det [[i, j]]_{n \times n}$ , where  $[i, j]$  is the least common multiple of  $i$  and  $j$ . In Example 5 of this paper we evaluate  $\det [[i, j]]_{n \times n}$  in a manner which shows that GCD and LCM matrices are, in a sense, similar in structure. The recent papers [3] and [8] also study LCM matrices. Further,

$\det [[i, j]]_{n \times n}$  has been presented in Problem 10232 of *Amer. Math. Monthly* 99(6) (1992) as a subject for evaluation. In Section 7 of this paper we consider a conjecture of Bourque and Ligh [8] on LCM matrices.

### 3. AN ELEMENTARY STRUCTURE THEOREM

Let  $(P, \leq)$  be a poset. We call  $P$  a meet semilattice [38, p. 103] if for any  $x, y \in P$  there exists a unique  $z \in P$  such that

- (1)  $z \leq x$  and  $z \leq y$ , and
- (2) if  $w \leq x$  and  $w \leq y$  for some  $w \in P$ , then  $w \leq z$ .

In such a case  $z$  is called the meet of  $x$  and  $y$  and is denoted by  $x \wedge y$ .

Let  $S$  be a subset of  $P$ . We call  $S$  lower-closed if for every  $x, y \in P$  with  $x \in S, y \leq x$ , we have  $y \in S$ . We call  $S$  meet-closed if for every  $x, y \in S$ , we have  $x \wedge y \in S$ . In this case  $S$  itself is a meet semilattice.

It is clear that a lower-closed subset of a meet semilattice is always meet-closed, but not conversely. The concepts of “lower-closed” and “meet-closed” are generalizations of “factor-closed” and “GCD-closed” [5, 6], respectively.

A function  $F$  on  $P \times P$  with values in a commutative ring with unit is said to be an incidence function of  $P$  if  $F(x, y) = 0$  unless  $x \leq y$  (see [1, Chapter IV; 38, Section 3.6]).

**THEOREM 1.** *Let  $P$  be a meet semilattice in which every principal order ideal is finite, and let  $\{x_1, x_2, \dots, x_n\}$  be a lower-closed subset of  $P$ . Let  $F$  and  $G$  be incidence functions of  $P$ , and let  $A$  be the  $n \times n$  matrix defined by*

$$a_{ij} = \sum_{x \leq x_i \wedge x_j} F(x, x_i)G(x, x_j).$$

Then

$$\det A = \prod_{k=1}^n F(x_k, x_k)G(x_k, x_k). \tag{3.1}$$

*Proof.* In evaluating  $\det A$  we may assume that the elements  $x_1, x_2, \dots, x_n$  are arranged so that  $x_i < x_j$  implies  $i < j$ . Since  $\{x_1, x_2, \dots, x_n\}$  is lower-closed,

$$a_{ij} = \sum_{k=1}^n F(x_k, x_i)G(x_k, x_j).$$

Let  $B$  and  $C$  be  $n \times n$  matrices such that

$$b_{ij} = F(x_i, x_j) \quad \text{and} \quad c_{ij} = G(x_i, x_j).$$

Then

$$A = B^T C,$$

and  $B$  and  $C$  are upper triangular matrices. We thus easily arrive at (3.1). ■

EXAMPLE 1. Let  $\zeta_D$  denote the zeta function of the poset  $(\mathbf{N}, |)$ , where  $\mathbf{N}$  is the set of positive integers. Then  $\zeta_D(d, m_i) = 1$  if  $d | m_i$ , and  $= 0$  otherwise. Let  $F$  and  $G$  be incidence functions of  $(\mathbf{N}, |)$  defined by  $F(d, m) = \zeta_D(d, m)$  and  $G(d, m) = \zeta_D(d, m)g(d)$ . Then (1.3) can be written as

$$f(m_i, m_j) = \sum_{d|(m_i, m_j)} F(d, m_i)G(d, m_j) = \sum_{k=1}^n F(m_k, m_i)G(m_k, m_j).$$

Application of Theorem 1 thus gives the evaluation (1.2).

EXAMPLE 2. Let  $\zeta_U$  denote the zeta function of the poset  $(\mathbf{N}, ||)$ . Then  $\zeta_U(d, i) = 1$  if  $d || i$ , and  $= 0$  otherwise. Let  $F$  and  $G$  be incidence functions of  $(\mathbf{N}, ||)$  defined by  $F(d, i) = \zeta_U(d, i)$  and  $G(d, j) = \zeta_U(d, j)g(d)$ . Then (2.3) can be written as

$$f(i, j) = \sum_{d||(i, j)^*} F(d, i)G(d, j) = \sum_{k=1}^n F(k, i)G(k, j).$$

Application of Theorem 1 thus gives the evaluation (2.4).

EXAMPLE 3. Denoting  $F(d, i) = \zeta_D(d, i)g(d)$  and  $G(d, j) = \zeta_D(d, j)h(j/d)$ , we can write (2.6) as

$$f(i, j) = \sum_{d|(i, j)} F(d, i)G(d, j) = \sum_{k=1}^n F(k, i)G(k, j),$$

where  $F$  and  $G$  are incidence functions of  $(\mathbf{N}, |)$ . Application of Theorem 1 thus gives the evaluation (2.7).

EXAMPLE 4. Let  $c^*(i, j)$  denote the unitary analogue of Ramanujan's sum [34, Section IX.4]. It is known [34, Section IX.4] that

$$c^*(i, j) = \sum_{\substack{d|i \\ d||j}} d\mu^*(j/d), \tag{3.2}$$

where  $\mu^*$  is the multiplicative function defined by  $\mu^*(1) = 1$  and  $\mu^*(p^r) = -1$  for all prime powers  $p^r (> 1)$ . Denoting  $F(d, i) = \zeta_D(d, i)d$  and  $G(d, j) = \zeta_V(d, j)\mu^*(j/d)$ , we can write (3.2) as

$$c^*(i, j) = \sum_{d|(i, j)} F(d, i)G(d, j) = \sum_{k=1}^n F(k, i)G(k, j),$$

where  $F$  and  $G$  are incidence functions of  $(\mathbf{N}, |)$ . Application of Theorem 1 thus gives

$$\det [c^*(i, j)]_{n \times n} = n!. \tag{3.3}$$

Note that

$$\det [c(i, j)]_{n \times n} = \det [c^*(i, j)]_{n \times n}.$$

EXAMPLE 5. Let  $g$  be the arithmetical function defined by

$$g(n) = \sum_{d|n} \frac{\mu(n/d)}{d}, \quad n = 1, 2, \dots,$$

where  $\mu$  is the Möbius function. By the Möbius inversion formula [28, Theorem 1.3] it follows that

$$\sum_{d|n} g(d) = \frac{1}{n}.$$

Thus

$$ij \sum_{d|(i, j)} g(d) = \frac{ij}{(i, j)} = [i, j],$$



where  $[i, j]$  is the least common multiple of  $i$  and  $j$ . Denoting  $F(d, i) = \zeta_D(d, i)i$  and  $G(d, j) = \zeta_D(d, j)g(d)j$ , we have

$$[i, j] = \sum_{d|(i, j)} F(d, i)G(d, j) = \sum_{k=1}^n F(k, i)G(k, j).$$

Here  $F$  and  $G$  are incidence functions of  $(\mathbf{N}, |)$ , and hence, by Theorem 1,

$$\det [[i, j]]_{n \times n} = \prod_{k=1}^n k^2 g(k). \tag{3.4}$$

It can be verified that

$$k^2 g(k) = \pi(k)\phi(k),$$

where  $\pi$  is the multiplicative function defined by  $\pi(1) = 1$  and  $\pi(p^r) = -p$  for all prime powers  $p^r (> 1)$ , and  $\phi$  is Euler's totient function. Thus

$$\det [[i, j]]_{n \times n} = \prod_{k=1}^n \pi(k)\phi(k). \tag{3.5}$$

Note that

$$\det [[i, j]]_{n \times n} = \det [(i, j)]_{n \times n} \prod_{k=1}^n \pi(k).$$

**EXAMPLE 6.** Let  $\zeta_N$  be the zeta function of  $(\mathbf{N}, \leq)$ , that is,  $\zeta_N(k, i) = 1$  if  $k \leq i$ , and  $= 0$  otherwise. Then

$$\min\{i, j\} = \sum_{k \leq \min\{i, j\}} \zeta_N(k, i)\zeta_N(k, j) = \sum_{k=1}^n \zeta_N(k, i)\zeta_N(k, j).$$

Thus, by Theorem 1,

$$\det [\min\{i, j\}]_{n \times n} = 1. \tag{3.6}$$

**REMARK.** It should be emphasized that, in all the above examples, matrices can be presented as the product of a lower and an upper triangular matrix. It is clear that these lower and upper triangular matrices are not unique.

## 4. A CONJECTURE OF BESLIN AND LIGH

Let  $P$  always denote a finite meet semilattice,  $S$  a poset that can be embedded in a meet semilattice, and  $\bar{S}$  the unique (up to isomorphism) minimal meet semilattice containing  $S$ .

Let  $S = \{x_1, x_2, \dots, x_n\}$  be a subset of  $P$ , and let  $f$  be a function on  $P$  with real values. Then the  $n \times n$  matrix  $(S)_f = (s_{ij})$ , where

$$s_{ij} = f(x_i \wedge x_j),$$

is called the meet matrix on  $S$  with respect to  $f$ .

The following Theorems 2 and 3 are generalizations of Beslin and Ligh's results [6, Theorem 1; 7, Theorem 1] about GCD matrices on GCD-closed and arbitrary sets of positive integers.

**THEOREM 2** [32]. *Let  $S = \{x_1, x_2, \dots, x_n\}$  be a meet-closed subset of  $P$ , and  $f$  a function on  $P$ . Then*

$$\det(S)_f = g(x_1)g(x_2) \cdots g(x_n),$$

where  $g(x_i)$  is defined by

$$g(x_i) = f(x_i) - \sum_{x_j \in S, x_j < x_i} g(x_j).$$

(Here  $x_j < x_i$  means that  $x_j \leq x_i$  and  $x_j \neq x_i$ .)

**COROLLARY** [26, 41]. *Let  $S = \{x_1, x_2, \dots, x_n\}$  be a lower-closed subset of  $P$ , and  $f$  a function on  $P$ . Then*

$$\det(S)_f = g(x_1)g(x_2) \cdots g(x_n),$$

where

$$g(x_i) = \sum_{x_j \leq x_i} f(x_j) \mu(x_j, x_i),$$

or equivalently

$$f(x_i) = \sum_{x_j \leq x_i} g(x_j),$$

$\mu$  being the Möbius function of  $P$  (see [38, p. 116]).

REMARK. The above corollary is a poset-theoretic generalization of Smith's result (1.2).

Let  $S = \{x_1, x_2, \dots, x_n\}$  and  $T = \{y_1, y_2, \dots, y_m\}$  be any subsets of  $P$ . Define the incidence matrix  $E(S, T)$  of  $S$  and  $T$  as an  $n \times m$  matrix whose  $i, j$ , entry is 1 if  $y_j \leq x_i$ , and 0 otherwise.

THEOREM 3 [32]. Let  $S = \{x_1, x_2, \dots, x_n\}$  be a subset of  $P$  with

$$\bar{S} = \{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+r}\}.$$

Let  $g$  be a function on  $\bar{S}$  defined as in Theorem 2. Then

$$(S)_f = E \operatorname{diag}(g(x_1), g(x_2), \dots, g(x_{n+r})) E^T,$$

where  $E = E(S, \bar{S})$  and  $E^T$  is the transpose of  $E$ .

REMARK. Theorem 3 is a generalization of (2.2).

By using a proof similar to that occurring in Li's paper for GCD matrices [23, Theorems 2 and 3] we have the following:

THEOREM 4. Let  $S, \bar{S}, f$ , and  $g$  be as in Theorem 3. Then

$$\det(S)_f = \sum_{1 \leq k_1 < k_2 < \dots < k_n \leq n+r} \det(E_{(k_1, k_2, \dots, k_n)})^2 g(x_{k_1}) g(x_{k_2}) \dots g(x_{k_n}),$$

where  $E_{(k_1, k_2, \dots, k_n)}$  is the submatrix of  $E = E(S, \bar{S})$  consisting of the  $k_1$ th,  $k_2$ th,  $\dots$ ,  $k_n$ th columns of  $E$ . Furthermore, if  $g$  is a function with positive values, then

$$\det(S)_f \geq g(x_1) g(x_2) \dots g(x_n),$$

and the equality holds if and only if  $S$  is meet-closed.

In the case in which  $S$  is a set of positive integers, Beslin and Ligh [7] proved that if  $S$  is GCD-closed or  $S$  is a  $k$ -set for some positive integer  $k$ , where a  $k$ -set is defined as a set of positive integers whose every pair of

distinct elements have the greatest common divisor  $k$ , then  $\det(E_{(k, k_2, \dots, k_n)}) = \pm 1$  for every choice of  $k_1, k_2, \dots, k_n$ . They also *conjectured* that the converse is true for  $n > 3$ . Li [24] shows that the converse does not hold for any  $n \geq 3$ .

For example, let  $n = 4$  and  $S = \{2, 3, 5, 6\}$ . Then  $\bar{S} = \{2, 3, 5, 6, 1\}$  and

$$E(S, \bar{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to verify that this is a counterexample of the conjecture.

In the next section we will give characterizations of the posets having a property similar to the example.

### 5. REGULAR POSETS

Beslin and Ligh's [7] conjecture and the example in the previous section raise the following question. What kind of posets  $S$  satisfy the property: If  $S$  has  $n$  elements, then the determinant of every  $n \times n$  submatrix of  $E(S, \bar{S})$  is equal to  $\pm 1$ ?

It is trivial that if  $S$  is a meet semilattice, then  $S$  satisfies this property. Otherwise we call  $S$  a *regular* poset.

**THEOREM 5.** *Let  $S = \{x_1, x_2, \dots, x_n\}$  be a poset with*

$$\bar{S} = \{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+r}\}, \quad r > 0.$$

*Then  $S$  is regular if and only if  $r = 1$ ,  $x_{n+1}$  is the minimum element of  $\bar{S}$ , and the system of linear equations*

$$E(S, S) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{5.1}$$

*has only the solution of the form  $(\pm 1, \pm 1, \dots, \pm 1)$ .*

*Proof.* Let  $E = E(S, \bar{S}) = (e_{ij})_{n \times (n+r)}$ . Then  $E(S, S)$  is the  $n \times n$  sub-matrix of  $E$ . For any  $i$ ,  $1 \leq i \leq r$ , consider the system of linear equations

$$E(S, S) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = E_{(n+i)}, \tag{5.2}$$

where  $E_{(n+i)}$  is the  $(n+i)$ th column of  $E$ . Let  $\sigma$  be a permutation of  $\{1, 2, \dots, n\}$  such that  $x_{\sigma(1)}$  is a minimal element of  $S$  and  $x_{\sigma(k)}$  a minimal element of  $S \setminus \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k-1)}\}$  for  $k = 2, 3, \dots, n$ . Then  $E(S, S)$  is similar to  $E(S', S')$ , where  $S' = \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}\}$ . It is easy to see that  $E(S', S')$  is a lower triangular matrix and  $\det E(S', S') = 1$ , which implies that  $\det E(S, S) = 1$ . By Cramer's rule we have

$$z_j = \det E_{(1, 2, \dots, j-1, n+i, j+1, \dots, n)}, \quad j = 1, 2, \dots, n. \tag{5.3}$$

From this it follows that if the conditions of the theorem hold, then  $S$  is regular.

Now, we suppose that  $S$  is regular. Then  $z_j = 1$  or  $-1$  as in (5.3). In order to complete the proof it suffices to show that  $x_{n+i}$  is the minimum element of  $\bar{S}$ , or  $E_{(n+i)}^T = (1, 1, \dots, 1)$ . Let  $x_j$  be any minimal element of  $S$ . Then the  $j$ th equation in (5.2) is  $z_j = e_{j, n+i}$ . Thus  $e_{j, n+i} = 1$ , from which it follows that  $x_{n+i} \leq x_j$ . The arbitrariness of  $x_j$  implies that  $x_{n+i} \leq x_k$  for  $k = 1, 2, \dots, n$ , i.e.,  $x_{n+i}$  is the minimum element of  $\bar{S}$ . Therefore  $r = 1$  and  $E_{(n+1)}^T = (1, 1, \dots, 1)$ . This completes the proof. ■

**COROLLARY 1.** Let  $S = \{x_1, x_2, \dots, x_n\}$  be a poset. Define a real function  $\nu_S$  as follows:

$$\nu_S(x) = \begin{cases} 1 & \text{if } x \text{ is a minimal element,} \\ 1 - \sum_{y < x} \nu_S(y) & \text{otherwise.} \end{cases}$$

Then  $S$  is regular if and only if  $\bar{S} = S \cup \{x_{n+1}\}$  and  $\nu_S(x) = \pm 1$  for every  $x \in S$ .

*Proof.* It is not difficult to verify that  $(\nu_S(x_1), \nu_S(x_2), \dots, \nu_S(x_n))^T$  is the solution of (5.1). Hence the conclusion follows from Theorem 5 at once. ■

COROLLARY 2. *Let  $S$  be a regular poset,  $x$  a maximal element of  $S$ . Then  $S \setminus \{x\}$  is also a regular poset.*

*Proof.* Write  $S \setminus \{x\} = T$ . By the definition of the function  $\nu$ , we see that  $\nu_T(y) = \nu_S(y)$  for every  $y \in T$ . The conclusion follows from Corollary 1. ■

The following theorem gives an inductive method to construct regular posets.

THEOREM 6.

- (i) *An incomparable set is regular.*
- (ii) *Suppose that  $T$  is a regular poset, and  $S = T \cup \{x\}$  is the union of disjoint sets  $T$  and  $\{x\}$  such that*
  - (1)  $x \not\leq y$  for every  $y \in T$ ,
  - (2)  $x$  covers the elements of an incomparable subset of  $T$ , say  $\{y_1, y_2, \dots, y_t\}$ , satisfying  $1 - \sum_y \nu_T(y) = \pm 1$ , where the sum is taken over all the elements  $y \in T$  with  $y \leq$  (one of  $y_1, y_2, \dots, y_t$ ), and
  - (3) *the poset  $S$  with the order defined as above can be embedded in a meet semilattice.*

*Then  $S$  is regular.*

It can be seen from Corollary 2 that, using the method in the above theorem, we can obtain all regular posets. However, it would be interesting to find a more effective algorithm.

Beslin and Ligh's conjecture is equivalent to saying that the incomparable sets are the only regular posets. In Figure 1 we list the Hasse diagrams of all regular posets with seven elements, except for the incomparable set.

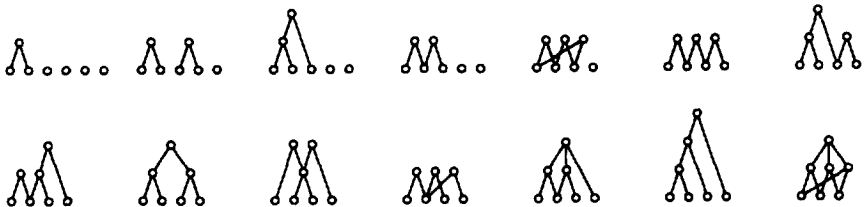


FIG. 1.

6. A-REGULAR POSETS

The concept of regular posets can be generalized as follows.

Let  $A$  be an arbitrary but fixed subset of  $\mathbf{Z}$  such that (1)  $0 \notin A$ , (2)  $1 \in A$ , and (3)  $a \in A$  implies  $-a \in A$ . We define a poset  $S$  of  $n$  elements to be *A-regular* if  $S$  is not a meet semilattice and the determinant value of every  $n \times n$  submatrix of  $E(S, \bar{S})$  belongs to  $A$ . Thus the regular posets are the *A-regular* posets with  $A = \{-1, 1\}$ .

It is easy to see that the proofs of our results for regular posets go through for *A-regular* posets. We thus obtain the following generalizations.

**THEOREM 7.** *Let  $S = \{x_1, x_2, \dots, x_n\}$  be a poset with*

$$\bar{S} = \{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+r}\}, \quad r > 0.$$

*Then  $S$  is A-regular if and only if  $r = 1$ ,  $x_{n+1}$  is the minimum element of  $\bar{S}$ , and the only solution of the system of linear equations*

$$E(S, S) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

*is of the form  $(z_1, z_2, \dots, z_n)$ , where  $z_1, z_2, \dots, z_n \in A$ .*

**COROLLARY 1.** *Let  $S = \{x_1, x_2, \dots, x_n\}$  be a poset. Define a real function  $\nu_S$  as follows:*

$$\nu_S(x) = \begin{cases} 1 & \text{if } x \text{ is a minimal element,} \\ 1 - \sum_{y < x} \nu_S(y) & \text{otherwise.} \end{cases}$$

*Then  $S$  is A-regular if and only if  $\bar{S} = S \cup \{x_{n+1}\}$  and  $\nu_S(x) \in A$  for every  $x \in S$ .*

**COROLLARY 2.** *Let  $S$  be an A-regular poset,  $x$  a maximal element of  $S$ . Then  $S \setminus \{x\}$  is also an A-regular poset.*

The following theorem gives an inductive method to construct  $A$ -regular posets.

**THEOREM 8.**

- (i) *An incomparable set is  $A$ -regular.*
- (ii) *Suppose that  $T$  is an  $A$ -regular poset, and  $S = T \cup \{x\}$  is the union of disjoint sets  $T$  and  $\{x\}$  such that*
  - (1)  $x \not\leq y$  for every  $y \in T$ ,
  - (2)  $x$  covers the elements of an incomparable subset of  $T$ , say  $\{y_1, y_2, \dots, y_t\}$ , satisfying  $1 - \sum_y \nu_T(y) \in A$ , where the sum is taken over all the elements  $y \in T$  with  $y \leq$  (one of  $y_1, y_2, \dots, y_t$ ), and
  - (3) the poset  $S$  with the order defined as above can be embedded in a meet semilattice.

*Then  $S$  is  $A$ -regular.*

It can be seen from Corollary 2 that, using the method in the above theorem, we can obtain all  $A$ -regular posets.

**REMARK.** If  $A = \mathbf{Z} \setminus \{0\}$ , then  $A$ -regular posets and meet semilattices give all the posets of  $n$  elements for which every  $n \times n$  submatrix of  $E(S, \bar{S})$  is invertible. For example, there are three  $A$ -regular posets of this type with four elements. These are the incomparable one and the following two posets shown in Figure 2. Note that the left poset corresponds to the example at the end of Section 4.

**7. A COUNTEREXAMPLE FOR A CONJECTURE OF BOURQUE AND LIGH**

Let  $S = \{x_1, x_2, \dots, x_n\}$  be a set of distinct positive integers. The matrix  $(S)$  having the greatest common divisor  $(x_i, x_j)$  as its  $i, j$  entry is called the GCD matrix on  $S$ . The matrix  $[S]$  having the least common multiple  $[x_i, x_j]$  as its  $i, j$  entry is called the LCM matrix on  $S$ . For further terminology and notation, see [8].



FIG. 2.



It is known that the GCD matrix on any set  $S$  is invertible [23, Theorem 3] and that there exist sets  $S$  such that the LCM matrix on  $S$  is not invertible [3, Remark 5]. It is also known that the LCM matrix on any factor-closed set is invertible [36, Section 3]. Further, it has been *conjectured* that the LCM matrix on any GCD-closed set is invertible; see Bourque and Ligh [8, p. 73]. We here show that this conjecture does not hold.

Let  $S = \{x_1, x_2, \dots, x_n\}$  be a GCD-closed set. Let  $g$  be the arithmetical function defined by

$$g(m) = \frac{1}{m} \sum_{d|m} d\mu(d) = \frac{\pi(m)\phi(m)}{m^2},$$

where  $\mu$  is the Möbius function,  $\phi$  is the Euler totient function and  $\pi$  is the multiplicative function such that  $\pi(p^r) = -p$  for all prime powers  $p^r$ . Bourque and Ligh [8, Theorem 5] show that

$$\det[S] = \prod_{i=1}^n x_i^2 \alpha_i, \quad \text{where } \alpha_i = \sum_{\substack{d|x_i \\ d \nmid x_t \\ x_t < x_i}} g(d). \quad (7.1)$$

Our calculations with the aid of the Mathematica system show that if  $x_i < 180$  for all  $i = 1, 2, \dots, n$ , then  $\det[S] \neq 0$ . However, if the greatest number in  $S$  is 180, then there exist GCD-closed sets  $S$  such that  $\det[S] = 0$ . For example, let

$$S = \{1, 2, 3, 4, 5, 6, 10, 45, 180\}. \quad (7.2)$$

Then  $S$  is GCD-closed but not factor-closed. Let  $x_n = x_9 = 180$ . Then

$$\begin{aligned} \alpha_n &= g(180) + g(90) + g(60) + g(36) + g(30) + g(20) + g(18) + g(12) \\ &= -\frac{2}{45} - \frac{4}{45} - \frac{2}{15} + \frac{1}{18} - \frac{4}{15} + \frac{1}{5} + \frac{1}{9} + \frac{1}{6} = 0, \end{aligned}$$

where  $\alpha_n$  is as given in (7.1). This shows that  $\det[S] = 0$  and thus (7.2) is a counterexample for the conjecture.

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