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On eigenvalues of meet and join matrices associated with incidence functions

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Abstract

Let $(P, \preceq, \wedge)$ be a locally finite meet semilattice. Let

$$S = \{x_1, x_2, \ldots, x_n\}, \quad x_i \preceq x_j \Rightarrow i \leqslant j,$$

be a finite subset of P and let f be a complex-valued function on P. Then the $n \times n$ matrix $(S)_f$, where

$$((S)_f)_{ij} = f(x_i \wedge x_j),$$

is called the meet matrix on S with respect to f. The join matrix on S with respect to f is defined dually on a locally finite join semilattice.

In this paper, we give lower bounds for the smallest eigenvalues of certain positive definite meet matrices with respect to f on any set S. We also estimate eigenvalues of meet matrices respect to any f on meet closed set S and with respect to semimultiplicative f on join closed set S. The same is carried out dually for join matrices.

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1. Introduction

Let $(P, \preceq)$ be a poset and let f be a complex-valued function on P. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of P such that $x_i \preceq x_j \Rightarrow i \leqslant j$. Throughout this paper we assume that S is nonempty and the elements of S are distinct. The poset P is said to be locally finite if the interval

$$[x, y] = \{z \in P | x \preceq z \preceq y\}$$

is finite for all $x, y \in P$. If the greatest lower bound of $x, y \in P$ exists, it is called the meet of x and y and is denoted by $x \wedge y$. If $x \wedge y \in P$ exists for all $x, y \in P$, then $(P, \preceq, \wedge)$ is called a meet semilattice. Let $(P, \preceq, \wedge)$ be a meet semilattice. Then the $n \times n$ matrix $(S)_f$, where $((S)_f)_{ij} = f(x_i \wedge x_j)$, is called the meet matrix on S with respect to f. If the least upper bound of $x, y \in P$ exists, it is called the join of x and y and is denoted by $x \vee y$. If $x \vee y \in P$ exists for all $x, y \in P$, then $(P, \preceq, \vee)$ is called a join semilattice. Let $(P, \preceq, \vee)$ be a join semilattice. Then the $n \times n$ matrix $[S]_f$, where $([S]_f)_{ij} = f(x_i \vee x_j)$, is called the join matrix on S with respect to f.

If the poset $(P, \preceq, \wedge, \vee)$ is both a meet semilattice and a join semilattice, it is called a lattice. The posets $(\mathbb{Z}_+, |)$ and $(\mathbb{Z}_+, \|)$, where $|$ is the divisibility relation and $\|$ is the unitary divisibility relation, are locally finite meet semilattices and the poset $(\mathbb{Z}_+, |)$ is also a locally finite lattice. Let S be a finite subset of $\mathbb{Z}_+$ and let f be a complex-valued function on $\mathbb{Z}_+$. Let (x_i, x_j) denote the greatest common divisor (GCD) of positive integers x_i and x_j and let $[x_i, x_j]$ denote the least common multiple (LCM) of positive integers x_i and x_j. The $n \times n$ matrix $(S)_f$, where $((S)_f)_{ij} = f((x_i, x_j))$, is called the GCD matrix on S with respect to f and the $n \times n$ matrix $[S]_f$, where $([S]_f)_{ij} = f([x_i, x_j])$, is called the LCM matrix on S with respect to f. The $n \times n$ matrix (S^α) having $(x_i, x_j)^\alpha$ as its ij entry is called the power GCD matrix on S. For $\alpha = 1$ we obtain the usual GCD matrix (S).

In 1876 Smith [18] calculated the determinant of the $n \times n$ matrix $((i, j))$, having the greatest common divisor of i and j as its ij entry. Since that lots of results concerning determinants and related topics of GCD matrices, LCM matrices, meet matrices and join matrices have been published in the literature. (See, for example [8,12,17].) Wintner [20] published results concerning the largest eigenvalue of the $n \times n$ matrix having

$$\frac{(i, j)^\alpha}{[i, j]^\alpha}$$

as its ij entry and subsequently Lindqvist and Seip [13] investigated the asymptotic behavior of the smallest and the largest eigenvalue of the same matrix. Beslin and Ligh [3] proved that the usual GCD matrices are positive definite and thus their eigenvalues are real and positive. Bourque and Ligh [6] extended this result by proving that for any $\alpha > 0$ the power GCD matrix is positive definite. Also Ovall [16] considered positive definiteness of GCD and related matrices. Balatoni [2] estimated the smallest and the largest eigenvalue of the $n \times n$ matrix $((i, j))$. Hong and Loewy [9] published results concerning the asymptotic behavior of eigenvalues of power GCD matrices. Recently, Bhatia [5] investigated infinitely divisible matrices and considered GCD matrices as an example.

In this paper, we consider the eigenvalues of meet and join matrices. There are no results published in the literature concerning the eigenvalues of meet and join matrices. We give a lower bound for the smallest eigenvalue of certain (real) positive definite meet and join matrices (see Sections 3 and 5). We adopt an argument similar to that used by Hong and Loewy [9, Theorem 4.2] to power GCD matrices. Our lattice-theoretic approach, however, makes it possible to consider also LCM matrices with the same method (and matrices with respect to f). Further we give a region in which all the eigenvalues of a complex meet matrix $(S)_f$ with respect to f on meet closed set S and with respect to semimultiplicative f on join closed set S lie (see Section 4). Dually we give a region in which all the eigenvalues of a complex join matrix $[S]_f$ with respect to f on join closed set S and with respect to semimultiplicative f on meet closed set S lie (see Sections 4 and 6). These results on complex meet and join matrices are new even for GCD and LCM matrices.

2. Preliminaries

A complex-valued function f on $P \times P$ such that $f(x, y) = 0$ whenever $x \npreceq y$ is called an incidence function of P. If f and g are incidence functions of P, their sum $f + g$ is defined by

$$(f + g)(x, y) = f(x, y) + g(x, y)$$

for all $x, y \in P$, their product fg is defined by

$$(fg)(x, y) = f(x, y)g(x, y)$$

for all $x, y \in P$ and their convolution $f * g$ is defined by

$$(f * g)(x, y) = \sum_{x \preceq z \preceq y} f(x, z)g(z, y)$$

for all $x, y \in P$. These functions are clearly incidence functions of P.

The incidence function δ of P defined by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise}, \end{cases}$$

is the unity under the convolution. The incidence function ζ of P is defined by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise}. \end{cases}$$

The inverse of ζ under the convolution is called the Möbius function of P and it is denoted by μ. For further material, see, for example [1,14,19].

We next review some preliminary results on meet matrices.

Let $(P, \preceq, \wedge, \hat{0})$ be a locally finite meet semilattice with the least element $\hat{0}$, that is, $\hat{0} \preceq x$ for all $x \in P$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite subset of P. The set S is said to be lower closed if $y \in S$ whenever $x \in S$, $y \in P$ with $y \preceq x$, and S is said to be meet closed if $x \wedge y \in S$ for all $x, y \in S$. The order ideal generated by S is defined as

$$\downarrow S = \{z \in P \mid \exists x \in S, z \preceq x\}.$$

Let $\downarrow S = \{w_1, w_2, \ldots, w_m\}$, with $w_i \preceq w_j \Rightarrow i \leqslant j$. Let f be a complex-valued function on P. We associate f with restricted incidence function f_d of $(P, \preceq, \wedge, \hat{0})$ by the formula

$$f_d(\hat{0}, z) = f(z), \ z \in P.$$

Proposition 2.1 [11, Lemma 3.2]. *Let $A = (a_{ij})$ denote the $n \times m$ matrix defined by*

$$a_{ij} = \begin{cases} \sqrt{(f_d * \mu)(\hat{0}, w_j)} & \textit{if } w_j \preceq x_i, \\ 0 & \textit{otherwise}. \end{cases}$$

Then $(S)_f = AA^{\mathrm{T}}$.

Proposition 2.2 [4, Theorem 12]. *Let S be meet closed and let E and $D = \mathrm{diag}(d_1, \ldots, d_n)$ denote the $n \times n$ matrices defined by*

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \preceq x_i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j, j<i}} (f_d * \mu)(\hat{0}, z).$$

Then $(S)_f = EDE^{\mathrm{T}}$.

Proposition 2.3 [7, Example 1]. *Let S be lower closed. Then*

$$f(x_i) = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j, j<i}} f(z), \quad x_i \in S.$$

We next review some preliminary results on join matrices.

Let $(P, \preceq, \vee, \hat{1})$ be a locally finite join semilattice with the greatest element $\hat{1}$, that is, $x \preceq \hat{1}$ for all $x \in P$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite subset of P. The set S is said to be upper closed if $y \in S$ whenever $x \in S$, $y \in P$ with $x \preceq y$, and S is said to be join closed if $x \vee y \in S$ for all $x, y \in S$. The dual order ideal generated by S is defined as

$$\uparrow S = \{z \in P \mid \exists x \in S, x \preceq z\}.$$

Let $\uparrow S = \{w_1, w_2, \ldots, w_m\}$, with $w_i \preceq w_j \Rightarrow i \leqslant j$. Let f be a complex-valued function on P. We associate f with restricted incidence function f_u of $(P, \preceq, \vee, \hat{1})$ by the formula

$$f_u(z, \hat{1}) = f(z), \quad z \in P.$$

Proposition 2.4 [12, Lemma 4.2]. *Let $A = (a_{ij})$ denote the $n \times m$ matrix defined by*

$$a_{ij} = \begin{cases} \sqrt{(\mu * f_u)(w_j, \hat{1})} & \text{if } x_i \preceq w_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[S]_f = AA^{\mathrm{T}}$.

Proposition 2.5. *Let S be join closed. Let E be the matrix defined in (2.1) and let $D = \mathrm{diag}(d_1, \ldots, d_n)$ denote the $n \times n$ matrix defined by*

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} (\mu * f_u)(z, \hat{1}).$$

Then $[S]_f = E^{\mathrm{T}} DE$.

Proposition 2.5 can be proved in a similar way to Proposition 2.2 or as a consequence of Proposition 2.4. For the sake of brevity we do not present the details.

Proposition 2.6 [12, Lemma 4.5]. *Let S be upper closed. Then*

$$f(x_i) = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} f(z), \quad x_i \in S.$$

We next review preliminary results on presenting certain meet matrices in terms of join matrices and certain join matrices in terms of meet matrices.

Let $(P, \preceq, \wedge, \vee)$ be a locally finite lattice. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite subset of P. Let f be a complex-valued function on P. We say that f is a semimultiplicative function if

$$f(x)f(y) = f(x \vee y)f(x \wedge y)$$

for all $x, y \in P$.

Proposition 2.7 [12, Lemma 5.2]. *Let f be a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$ and let $D = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. Then*

$$(S)_f = D[S]_{1/f} D.$$

Proposition 2.8 [12, Lemma 5.1]. *Let f be a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$ and let $D = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. Then*

$$[S]_f = D(S)_{1/f} D.$$

Let $K(n)$ denote the set of all $n \times n$ lower triangular 0,1 matrices such that each main diagonal entry is equal to 1. Clearly every matrix $X \in K(n)$ is real and nonsingular and thus XX^{T} is positive definite. Now we define the positive constants c_n [9] and C_n depending only on n such that

$$c_n = \min\{\lambda | X \in K(n), \lambda \text{ is the smallest eigenvalue of } XX^{\mathrm{T}}\}$$

and

$$C_n = \max\{\lambda | X \in K(n), \lambda \text{ is the largest eigenvalue of } XX^{\mathrm{T}}\}.$$

In Sections 3–6, we use the constants c_n and C_n in estimating eigenvalues of certain meet and join matrices. We estimate these constants themselves in Section 7.

3. Lower bound for the smallest eigenvalue of certain positive definite meet matrices

In this section, we provide a lower bound for the smallest eigenvalue of certain positive definite meet matrices with respect to f on any finite subset of P. As examples we consider GCUD (greatest common unitary divisor) and GCD matrices. Eigenvalues of meet matrices and GCUD matrices have not hitherto been studied in the literature.

Theorem 3.1. *Let $(P, \preceq, \wedge, \hat{0})$ be a locally finite meet semilattice that has the least element $\hat{0}$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite subset of P and let $\downarrow S = \{w_1, w_2, \ldots, w_m\}$, with $w_i \preceq w_j \Rightarrow i \leqslant j$. Let f be a real-valued function on P. Let $\lambda_1(n)$ denote the smallest eigenvalue of the matrix $(S)_f$. If*

$$(f_d * \mu)(\hat{0}, w_i) > 0 \quad \textit{for all } w_i \in \downarrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1 \leqslant i \leqslant n} (f_d * \mu)(\hat{0}, x_i).$$

Proof. Let $A = (a_{ij})$ denote the $n \times m$ matrix defined by

$$a_{ij} = \begin{cases} \sqrt{(f_d * \mu)(\hat{0}, w_j)} & \text{if } w_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Proposition 2.1 that $(S)_f = AA^{\mathrm{T}}$. We can permute the columns of A with any permutation matrix Q and $AA^{\mathrm{T}} = (AQ)(AQ)^{\mathrm{T}}$, so we may assume without loss of generality that

$$w_i = x_i, \quad 1 \leqslant i \leqslant n.$$

The matrix A can be partitioned as

$$A = [B|C],$$

where B is an $n \times n$ matrix and C is an $(m - n) \times n$ matrix. Now

$$AA^{\mathrm{T}} = [B|C][B|C]^{\mathrm{T}} = [B|C]\left[\frac{B^{\mathrm{T}}}{C^{\mathrm{T}}}\right] = BB^{\mathrm{T}} + CC^{\mathrm{T}}.$$

Let $\mu_1(n)$ denote the smallest eigenvalue of the matrix BB^{T}. Since

$$CC^{\mathrm{T}} = AA^{\mathrm{T}} - BB^{\mathrm{T}}$$

and the matrix CC^{T} is positive semidefinite, we have (see, for example [10, p. 471])

$$\lambda_1(n) \geqslant \mu_1(n).$$

Now, consider the $n \times n$ matrix $B = (b_{ij})$. We have

$$b_{ij} = \begin{cases} \sqrt{(f_d * \mu)(\hat{0}, x_j)} & \text{if } x_j \preceq x_i, \\ 0 & \text{otherwise,} \end{cases}$$

and thus the matrix B can be written as

$$B = ED,$$

where E is the matrix defined in (2.1) and $D = \mathrm{diag}(d_1, \ldots, d_n)$ with

$$d_i = \sqrt{(f_d * \mu)(\hat{0}, x_i)}.$$

Now, we use the spectral norm which we denote by $\|\cdot\|$. The matrix BB^{T} is positive definite and thus the inverse B^{-1} exists and the largest eigenvalue of the matrix $(BB^{\mathrm{T}})^{-1}$ is $\|(BB^{\mathrm{T}})^{-1}\|$. We have

$$\begin{aligned}\|(D^2)^{-1}\| &= \left\|\mathrm{diag}\left(\frac{1}{(f_d * \mu)(\hat{0}, x_1)}, \ldots, \frac{1}{(f_d * \mu)(\hat{0}, x_n)}\right)\right\| \\ &= \max_{1 \leqslant i \leqslant n} \frac{1}{(f_d * \mu)(\hat{0}, x_i)} = \frac{1}{\min_{1 \leqslant i \leqslant n}(f_d * \mu)(\hat{0}, x_i)}\end{aligned}$$

and since

$$\|MM^{\mathrm{T}}\| = \|M\| \cdot \|M^{\mathrm{T}}\| = \|M\|^2$$

for any square matrix M, we have

$$\begin{aligned}\|(BB^{\mathrm{T}})^{-1}\| &= \|(ED(ED)^{\mathrm{T}})^{-1}\| = \|(E^{\mathrm{T}})^{-1}(D^2)^{-1}E^{-1}\| \\ &\leqslant \|(E^{\mathrm{T}})^{-1}\| \cdot \|(D^2)^{-1}\| \cdot \|E^{-1}\| = \|(D^2)^{-1}\| \cdot \|(EE^{\mathrm{T}})^{-1}\|.\end{aligned}$$

Clearly, the matrix E belongs to the set $K(n)$ defined in Section 2 and hence

$$\|(EE^{\mathrm{T}})^{-1}\| \leqslant \frac{1}{c_n}.$$

We conclude that

$$\lambda_1(n) \geqslant \mu_1(n) = \frac{1}{\|(BB^{\mathrm{T}})^{-1}\|} \geqslant c_n \cdot \min_{1\leqslant i\leqslant n} (f_d * \mu)(\hat{0}, x_i). \qquad \square$$

Example 3.1. Let $(P, \preceq) = (\mathbb{Z}_+, \|)$, where $\|$ denotes the unitary divisibility relation defined by $d\|x$ if $d|x$ and $(d, x/d) = 1$. The greatest lower bound of $x_i, x_j \in \mathbb{Z}_+$ is their greatest common unitary divisor

$$x_i \wedge x_j = (x_i, x_j)^{**}.$$

Now, $(\mathbb{Z}_+, \|)$ is a locally finite meet semilattice possessing the least element $1 \in \mathbb{Z}_+$.

The unitary convolution of two arithmetical functions f and g is defined by

$$(f *_U g)(n) = \sum_{d\|n} f(d)g\left(\frac{n}{d}\right)$$

and the arithmetical function δ defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is the identity under the unitary convolution. Let $\zeta(n) = 1$ for all positive integers n. The unitary analogue μ^* of the Möbius function is the inverse of ζ under the unitary convolution. The unitary analogue μ^* of the Möbius function can be written as $\mu^*(1) = 1$ and $\mu^*(n) = (-1)^{w(n)}$ for $n > 1$, where $w(n)$ is the number of distinct prime divisors of n.

Now, let $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}_+$ be finite and

$$\downarrow S = \{w_1, w_2, \ldots, w_m\}, \quad w_i \| w_j \Rightarrow i \leqslant j.$$

Let f be an arithmetical function. Let $\lambda_1(n)$ denote the smallest eigenvalue of the matrix $(S^{**})_f$ having

$$f((x_i, x_j)^{**})$$

as its ij entry. Since the least element of $(\mathbb{Z}_+, \|)$ is 1, we have

$$f_d(\hat{0}, z) = f_d(1, z) = f(z).$$

Now

$$(f_d * \mu)(1, x) = \sum_{1\|y\|x} f_d(1, y)\mu(y, x) = \sum_{y\|x} f(y)\mu(y, x).$$

Since

$$\zeta(y, x) = \zeta\left(\frac{x}{y}\right) \quad \text{for } y\|x$$

and

$$\delta(y, x) = \delta\left(\frac{x}{y}\right) \quad \text{for } y\|x,$$

we have

$$\sum_{y\|x} f(y)\mu(y,x) = \sum_{y\|x} f(y)\mu^*\left(\frac{x}{y}\right) = (f *_U \mu^*)(x).$$

Now it follows from Theorem 3.1 that if

$$(f *_U \mu^*)(w_i) > 0 \quad \text{for all } w_i \in\downarrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1\leqslant i\leqslant n} (f *_U \mu^*)(x_i).$$

For instance, if $f(n) = n^\alpha$, where $\alpha \in \mathbb{R}_+$, then $(S^{**})_f$ may be referred as the power GCUD matrix on S and $(f *_U \mu^*)(n) = J_\alpha^*(n) > 0$ for all $n \in \mathbb{Z}_+$, where J_α^* is the unitary analogue of the Jordan totient function. For $\alpha = 1$, J_α^* is the unitary analogue of the Euler totient function. For estimations of values of the Jordan totient function and its unitary analogue, see [15].

Example 3.2. Let $(P, \preceq) = (\mathbb{Z}_+, |)$. Now, the greatest lower bound of $x_i, x_j \in \mathbb{Z}_+$ is their greatest common divisor

$$x_i \wedge x_j = (x_i, x_j).$$

Let $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}_+$ be finite and let f be an arithmetical function and let μ denote the number-theoretic Möbius function.

We can easily show (as in Example 3.1) using Theorem 3.1 that if

$$(f * \mu)(w_i) > 0 \quad \text{for all } w_i \in\downarrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1\leqslant i\leqslant n} (f * \mu)(x_i),$$

where $*$ is the Dirichlet convolution.

We want to remind that Hong and Loewy [9] have already covered the case $f(n) = n^\alpha$, where $\alpha \in \mathbb{R}_+$, of this example.

4. On eigenvalues of meet matrices with respect to f on meet closed sets

All published results concerning eigenvalues of GCD and related matrices have dealt with *real* (symmetric) matrices. The following theorem is the first attempt to estimate eigenvalues of a meet matrix that is *complex* (and symmetric). All the eigenvalues of a real symmetric matrix are real but this is not the case for complex symmetric matrices. We here consider meet matrices with respect to any f on meet closed sets. As a corollary we obtain dual results for join matrices with respect to semimultiplicative f on meet closed sets.

Theorem 4.1. *Let $(P, \preceq, \wedge, \hat{0})$ be a locally finite meet semilattice that has the least element $\hat{0}$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite meet closed subset of P. Let f be any complex-valued function on P. Then every eigenvalue of the matrix $(S)_f$ lies in the region*

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} : |z - f(x_k)| \leqslant C_n \cdot \max_{1\leqslant i\leqslant n} |d_i| - |f(x_k)| \right\},$$

where

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j, j<i}} (f_d * \mu)(\hat{0}, z).$$

Proof. Let E denote the matrix defined in (2.1) and let $D = \mathrm{diag}(d_1, \dots, d_n)$, where

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j, j<i}} (f_d * \mu)(\hat{0}, z).$$

It follows from Proposition 2.2 that $(S)_f = EDE^{\mathrm{T}}$. Let $|A|$ denote the matrix of the absolute values of the entries of the matrix A. We have

$$|(S)_f| = |EDE^{\mathrm{T}}| \leqslant E|D|E^{\mathrm{T}},$$

where $\leqslant$ is understood entrywise.

The matrix $|D|$ can be written as

$$|D| = \Lambda\Lambda^{\mathrm{T}},$$

where $\Lambda^{\mathrm{T}} = \Lambda = \mathrm{diag}(l_1, \dots, l_n)$ is the $n \times n$ matrix defined by

$$l_i = \sqrt{\left| \sum_{\substack{z \preceq x_i \\ z \npreceq x_j, j<i}} (f_d * \mu)(\hat{0}, z) \right|}.$$

The matrix

$$E\Lambda(E\Lambda)^{\mathrm{T}} = E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}$$

is positive semidefinite and thus its spectral radius is

$$\rho(E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}) = \|E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}\|.$$

Now, we have

$$\|E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}\| \leqslant \|E\| \cdot \|\Lambda\Lambda^{\mathrm{T}}\| \cdot \|E^{\mathrm{T}}\| = \|EE^{\mathrm{T}}\| \cdot \|\Lambda\Lambda^{\mathrm{T}}\|,$$

and since the matrix E belongs to the set $K(n)$ defined in Section 2, we have

$$\|EE^{\mathrm{T}}\| \leqslant C_n.$$

Since

$$\|\Lambda\Lambda^{\mathrm{T}}\| = \max_{1 \leqslant i \leqslant n} |d_i|,$$

it follows that

$$\rho(E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}) \leqslant C_n \cdot \max_{1 \leqslant i \leqslant n} |d_i|.$$

It is known (see, for example [10, p. 501]) that if A and B are $n \times n$ matrices such that the matrix B has nonnegative entries and $B \geqslant |A|$, then every eigenvalue of the matrix A lies in the region

$$\bigcup_{k=1}^{n}\{z \in \mathbb{C}: |z - a_{kk}| \leqslant \rho(B) - b_{kk}\}.$$

Let $A = (S)_f$ and $B = E|D|E^{\mathrm{T}} = E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}$. Since we have

$$\rho(E\Lambda\Lambda^{\mathrm{T}}E^{\mathrm{T}}) - b_{kk} \leqslant C_n \cdot \max_{1\leqslant i\leqslant n} |d_i| - |f(x_k \wedge x_k)| = C_n \cdot \max_{1\leqslant i\leqslant n} |d_i| - |f(x_k)|,$$

we conclude that every eigenvalue of the matrix $(S)_f$ lies in the region

$$\bigcup_{k=1}^{n}\{z \in \mathbb{C}: |z - f(x_k)| \leqslant C_n \cdot \max_{1\leqslant i\leqslant n} |d_i| - |f(x_k)|\}. \qquad \square$$

Obviously Theorem 4.1 may also be used to find an upper bound for the largest eigenvalue of the meet matrix $(S)_f$ with respect to a real f on meet closed set S.

Remark 4.1. If the set S is lower closed, then it follows from Proposition 2.3 that

$$\sum_{\substack{z\preceq x_i \\ z\npreceq x_j, j<i}} (f_d * \mu)(\hat{0}, z) = (f_d * \mu)(\hat{0}, x_i)$$

and hence in Theorem 4.1 we have

$$\max_{1\leqslant i\leqslant n} |d_i| = \max_{1\leqslant i\leqslant n} |(f_d * \mu)(\hat{0}, x_i)|.$$

Example 4.1. Let $(P, \preceq, \wedge) = (\mathbb{Z}_+, |, \mathrm{GCD})$ and let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i|x_j \Rightarrow i \leqslant j$, be a finite lower closed subset of $\mathbb{Z}_+$. Let $\alpha \in \mathbb{C}$. Let $f(n) = n^\alpha$ for all $n \in \mathbb{Z}_+$, where n^α means the principal value of the complex power. Now

$$(f * \mu)(x_i) = J_\alpha(x_i),$$

where J_α is a complex generalization of the Jordan totient function, and it follows from Theorem 4.1 that every eigenvalue of the matrix $(S)_f$ lies in the region

$$\bigcup_{k=1}^{n}\left\{z \in \mathbb{C}: |z - x_k^\alpha| \leqslant C_n \cdot \max_{1\leqslant i\leqslant n} |J_\alpha(x_i)| - x_k^{\mathrm{Re}(\alpha)}\right\}.$$

For $\alpha = 1$

$$(f * \mu)(x_i) = \varphi(x_i),$$

where φ is the Euler totient function, and every eigenvalue of the matrix $(S)_f$ lies in the set

$$\bigcup_{k=1}^{n}\left\{z \in \mathbb{R}: |z - x_k| \leqslant C_n \cdot \max_{1\leqslant i\leqslant n} \varphi(x_i) - x_k\right\}.$$

The following corollary concerns join matrices on meet closed sets.

Corollary 4.1. *Let $(P, \preceq, \wedge, \vee, \hat{0})$ be a locally finite lattice that has the least element $\hat{0}$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite meet closed subset of P. Let f be a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$. Define the function g on P by $g(x) = \frac{1}{f(x)}$ for all $x \in P$. Then every eigenvalue of the matrix $[S]_f$ lies in the region*

$$\bigcup_{k=1}^{n}\left\{z \in \mathbb{C}\colon |z - f(x_k)| \leqslant \max_{1\leqslant i\leqslant n} f^2(x_i)\cdot C_n \cdot \max_{1\leqslant i\leqslant n} |d_i| - |f(x_k)|\right\},$$

where

$$d_i = \sum_{\substack{z\preceq x_i \\ z \npreceq x_j, j<i}} (g_d * \mu)(\hat{0}, z).$$

Proof. It follows from Proposition 2.8 that

$$[S]_f = D(S)_g D,$$

where $D = \mathrm{diag}(f(x_1), \ldots, f(x_n))$. Since $|D(S)_g D| \leqslant |D|\cdot|(S)_g|\cdot|D|$ and since

$$\rho(|D|\cdot|(S)_g|\cdot|D|) = \|\,|D|\cdot|(S)_g|\cdot|D|\,\| \leqslant \max_{1\leqslant i\leqslant n} f^2(x_i)\cdot\|(S)_g\|,$$

the result follows from the proof of Theorem 4.1. □

5. Lower bound for the smallest eigenvalue of certain positive definite join matrices

In this section we convert Theorem 3.1 on meet matrices into the setting of join matrices, that is, we provide a lower bound for the smallest eigenvalue of certain positive definite join matrices with respect to f on any finite subset of P. As an example we consider LCM matrices. We do not examine LCUM matrices here, since LCUM does not always exist. We will study this topic in another paper.

Theorem 5.1. *Let* $(P, \preceq, \vee, \hat{1})$ *be a locally finite join semilattice that has the greatest element* $\hat{1}$. *Let* $S = \{x_1, x_2, \ldots, x_n\}$, *with* $x_i \preceq x_j \Rightarrow i \leqslant j$, *be a finite subset of* P *and let* $\uparrow S = \{w_1, w_2, \ldots, w_m\}$, *with* $w_i \preceq w_j \Rightarrow i \leqslant j$. *Let* f *be a real-valued function on* P. *Let* $\lambda_1(n)$ *denote the smallest eigenvalue of the matrix* $[S]_f$. *If*

$$(\mu * f_u)(w_i, \hat{1}) > 0 \quad \text{for all } w_i \in\uparrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1\leqslant i\leqslant n} (\mu * f_u)(x_i, \hat{1}).$$

Proof. Let $A = (a_{ij})$ denote the $n \times m$ matrix defined by

$$a_{ij} = \begin{cases} \sqrt{(\mu * f_u)(w_j, \hat{1})} & \text{if } x_i \preceq w_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from Proposition 2.4 that $[S]_f = AA^{\mathrm{T}}$. We may assume without loss of generality that

$$w_i = x_i, \quad 1 \leqslant i \leqslant n.$$

The matrix A can be partitioned as

$$A = [B|C],$$

where B is an $n \times n$ matrix and C is an $(m-n) \times n$ matrix. Now

$$AA^{\mathrm{T}} = BB^{\mathrm{T}} + CC^{\mathrm{T}}.$$

Let $\mu_1(n)$ be the smallest eigenvalue of the matrix BB^{T}. We have

$$\lambda_1(n) \geqslant \mu_1(n).$$

Consider now the $n \times n$ matrix $B = (b_{ij})$. We have

$$b_{ij} = \begin{cases} \sqrt{(\mu * f_u)(x_j, \hat{1})} & \text{if } x_i \preceq x_j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix B can be written as

$$B = E^{\mathrm{T}} D,$$

where E is the matrix defined in (2.1) and $D = \operatorname{diag}(d_1, \ldots, d_n)$ such that

$$d_i = \sqrt{(\mu * f_u)(x_i, \hat{1})}.$$

We have

$$\|(D^2)^{-1}\| = \frac{1}{\min_{1\leqslant i\leqslant n}(\mu * f_u)(x_i, \hat{1})}$$

and since

$$\|(E^{\mathrm{T}}E)^{-1}\| = \|(EE^{\mathrm{T}})^{-1}\|,$$

we have

$$\|(BB^{\mathrm{T}})^{-1}\| \leqslant \|(D^2)^{-1}\| \cdot \|(EE^{\mathrm{T}})^{-1}\| \leqslant \|(D^2)^{-1}\| \cdot \frac{1}{c_n}.$$

We conclude that

$$\lambda_1(n) \geqslant \mu_1(n) = \frac{1}{\|(BB^{\mathrm{T}})^{-1}\|} \geqslant c_n \cdot \min_{1\leqslant i\leqslant n} (\mu * f_u)(x_i, \hat{1}). \qquad \square$$

It is not as easy to utilize results on eigenvalues of join matrices to eigenvalues of LCM matrices as to utilize results on eigenvalues of meet matrices to eigenvalues of GCD matrices. The problem is that there does not exist the greatest element in $\mathbb{Z}_+$. Korkee and Haukkanen [12, p. 54], however, have found a way to transfer their results on determinants of join matrices to determinants of LCM matrices. In the following example we use an approach similar to that when we apply Theorem 5.1 to LCM matrices.

Example 5.1. Let $(P, \preceq) = (\mathbb{Z}_+, |)$. Now, the least upper bound of $x_i, x_j \in \mathbb{Z}_+$ is their least common multiple

$$x_i \vee x_j = [x_i, x_j].$$

Let $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}_+$ be finite and let f be an arithmetical function. Let

$$s = [x_1, x_2, \ldots, x_n]$$

denote the LCM of $x_1, x_2, \ldots, x_n$ and let T_s be the set of all positive divisors of s. Now, $(T_s, |, \mathrm{LCM}, s)$ is a locally finite join semilattice with the greatest element s such that $x|s$ for all $x \in T_s$, and S is a finite subset of T_s.

Let $\lambda_1(n)$ denote the smallest eigenvalue of the matrix $[S]_f$. Now it follows from Theorem 5.1 that if

$$(\mu * f_u)(w_i, s) > 0 \quad \text{for all } w_i \in \uparrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1 \leqslant i \leqslant n} (\mu * f_u)(x_i, s).$$

We have

$$(\mu * f_u)(z, s) = \sum_{z|y|s} \mu(y/z) f(y),$$

where μ on the right-hand side of the equation above is the number-theoretic Möbius function. Thus if

$$\sum_{w_i|y|s} \mu(y/w_i) f(y) > 0 \quad \text{for all } w_i \in \uparrow S,$$

then

$$\lambda_1(n) \geqslant c_n \cdot \min_{1 \leqslant i \leqslant n} \sum_{x_i|y|s} \mu(y/x_i) f(y).$$

6. On eigenvalues of join matrices with respect to f on join closed sets

In this section, we go through the results on meet matrices given in Section 4 dually for join matrices. As a corollary we obtain dual results for meet matrices with respect to semimultiplicative f on join closed sets. The results of this section are new even in $(\mathbb{Z}_+, |)$.

Theorem 6.1. *Let $(P, \preceq, \vee, \hat{1})$ be a locally finite join semilattice that has the greatest element $\hat{1}$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite join closed subset of P. Let f be any complex-valued function on P. Then every eigenvalue of the matrix $[S]_f$ lies in the region*

$$\bigcup_{k=1}^{n} \{z \in \mathbb{C}: |z - f(x_k)| \leqslant C_n \cdot \max_{1 \leqslant i \leqslant n} |d_i| - |f(x_k)|\},$$

where

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} (\mu * f_u)(z, \hat{1}).$$

Proof. Let E denote the matrix defined in (2.1) and let $D = \mathrm{diag}(d_1, \ldots, d_n)$, where

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} (\mu * f_u)(z, \hat{1}).$$

Then it follows from Proposition 2.5 that $[S]_f = E^{\mathrm{T}} D E$. We have

$$|[S]_f| = |E^{\mathrm{T}} D E| \leqslant E^{\mathrm{T}} |D| E.$$

The matrix $|D|$ can be written as

$$|D| = \Lambda\Lambda^{\mathrm{T}},$$

where $\Lambda^{\mathrm{T}} = \Lambda = \mathrm{diag}(l_1, \ldots, l_n)$ is the $n \times n$ matrix defined by

$$l_i = \sqrt{\left|\sum_{x_i \preceq z, x_j \npreceq z, i<j} (\mu * f_u)(z, \hat{1})\right|}.$$

We have

$$\rho(E^{\mathrm{T}}\Lambda\Lambda^{\mathrm{T}}E) \leqslant \|EE^{\mathrm{T}}\| \cdot \|\Lambda\Lambda^{\mathrm{T}}\|$$

and

$$\|EE^{\mathrm{T}}\| \leqslant C_n.$$

Since

$$\|\Lambda\Lambda^{\mathrm{T}}\| = \max_{1 \leqslant i \leqslant n} |d_i|,$$

it follows that

$$\rho(E^{\mathrm{T}}\Lambda\Lambda^{\mathrm{T}}E) \leqslant C_n \cdot \max_{1 \leqslant i \leqslant n} |d_i|.$$

We conclude that every eigenvalue of the matrix $[S]_f$ lies in the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \colon |z - f(x_k)| \leqslant C_n \cdot \max_{1 \leqslant i \leqslant n} |d_i| - |f(x_k)| \right\}. \qquad \square$$

Remark 6.1. If the set S is upper closed, then it follows from Proposition 2.6 that

$$\sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} (\mu * f_u)(z, \hat{1}) = (\mu * f_u)(x_i, \hat{1})$$

and hence in Theorem 4.1 we have

$$\max_{1 \leqslant i \leqslant n} |d_i| = \max_{1 \leqslant i \leqslant n} |(\mu * f_u)(x_i, \hat{1})|.$$

The following corollary concerning meet matrices on join closed sets follows from Proposition 2.7 and Theorem 6.1.

Corollary 6.1. *Let $(P, \preceq, \wedge, \vee, \hat{1})$ be a locally finite lattice that has the greatest element $\hat{1}$. Let $S = \{x_1, x_2, \ldots, x_n\}$, with $x_i \preceq x_j \Rightarrow i \leqslant j$, be a finite join closed subset of P. Let f be a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$. Define the function g on P by $g(x) = \frac{1}{f(x)}$ for all $x \in P$. Then every eigenvalue of the matrix $(S)_f$ lies in the region*

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \colon |z - f(x_k)| \leqslant \max_{1 \leqslant i \leqslant n} f^2(x_i) \cdot C_n \cdot \max_{1 \leqslant i \leqslant n} |d_i| - |f(x_k)| \right\},$$

where

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z, i<j}} (\mu * g_u)(z, \hat{1}).$$

In this article, we concentrated on the eigenvalues of meet and join matrices. It would be possible to investigate the eigenvalues of other related matrices, for example reciprocal matrices $f(x_i \wedge x_j)/f(x_i \vee x_j)$, by using the same methods.

7. Estimating c_n and C_n

In numerical computations our results require to know a lower bound for c_n and an upper bound for C_n. We study first the latter question.

Let $X \in K(n)$. The largest eigenvalue of $M = XX^{\mathrm{T}}$, equal to $\rho(M)$, the spectral radius of M, is increasing with respect to the entries of M (see [10, Corollary 8.1.19]) and so also with respect to the entries of X. Therefore, $C_n = \rho(M_0)$, where $M_0 = X_0X_0^{\mathrm{T}}$ and X_0 has all the lower triangle entries equal to 1. As an upper bound for the spectral radius of

$$M_0 = X_0X_0^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix},$$

we apply the Frobenius norm

$$\begin{aligned} T_n &= \|M_0\|_F = (\operatorname{tr} M_0^2)^{\frac{1}{2}} \\ &= [(2n-1) + (2n-3)\cdot 4 + (2n-5)\cdot 9 + \cdots + 3\cdot(n-1)^2 + n^2]^{\frac{1}{2}}. \end{aligned}$$

This upper bound seems to be pretty good. For example, we have $C_3 = 5.04892$, $T_3 = 5.09902$, $C_5 = 12.3435$, $T_5 = 12.4499$, and $C_9 = 36.6604$, $T_9 = 36.9459$.

We obtain a still better bound by a suitable shifting. Choose $t \leqslant 1$ smartly and compute $T_n(t) = t + \|M_0 - tI\|_F$. (So $T_n(0) = T_n$.) For example, we have $T_3(0.5) = 5.05522$, $T_5(0.7) = 12.3812$, and $T_9(1) = 36.8329$.

The question of presenting a lower bound for c_n is harder since it seems difficult to find X_0 such that $M_0 = X_0X_0^{\mathrm{T}}$ satisfies $c_n = \lambda(M_0)$, where λ denotes the smallest eigenvalue of M_0. We, however, propose the following conjecture.

Conjecture 7.1. *Let $X_0 = (x_{ij}^0)$ be defined by*

$$x_{ij}^0 = \begin{cases} 0 & \text{if } i > j \text{ and } i+j \text{ is even,} \\ 1 & \text{if } i > j \text{ and } i+j \text{ is odd.} \end{cases}$$

Then $c_n = \lambda(X_0X_0^{\mathrm{T}})$, where λ denotes the smallest eigenvalue of $X_0X_0^{\mathrm{T}}$.

Calculations show that this conjecture holds for $n = 2, 3, \ldots, 7$.

Note that the constant c_n was already introduced by Hong and Loewy [9]. Therefore, analyzing c_n provides further information also for their paper.

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