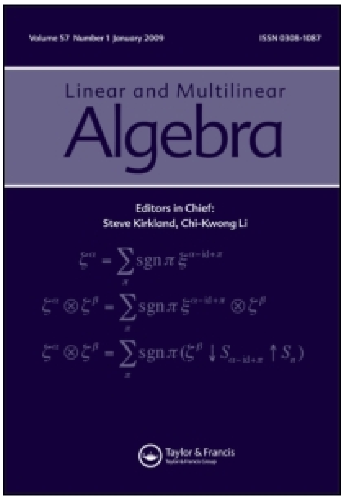

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On unitary analogs of GCD reciprocal LCM matrices

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A divisor $d \in \mathbb{Z}_+$ of $n \in \mathbb{Z}_+$ is said to be a unitary divisor of n if $(d, n/d) = 1$. In this article we examine the greatest common unitary divisor (GCUD) reciprocal least common unitary multiple (LCUM) matrices. At first we concentrate on the difficulty of the non-existence of the LCUM and we present three different ways to overcome this difficulty. After that we calculate the determinant of the three GCUD reciprocal LCUM matrices with respect to certain types of functions arising from the LCUM problematics. We also analyse these classes of functions, which may be referred to as unitary analogs of the class of semimultiplicative functions, and find their connections to rational arithmetical functions. Our study shows that it does make a difference how to extend the concept of LCUM.

Keywords: GCD matrix; LCM matrix; unitary divisor meet semilattice; semimultiplicative function; rational arithmetical function

AMS Subject Classifications: 06A12; 11A25; 11C20; 15A36

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and let f be an arithmetical function. The greatest common divisor (GCD) matrix $(S)_f$ on S with respect to f is defined as the $n \times n$ matrix whose ij entry is f evaluated at the GCD of x_i and x_j, that is, the ij entry of $(S)_f$ is $f((x_i, x_j))$. The least common multiple (LCM) matrix $[S]_f$ on S with respect to f is defined analogously.

In 1876 Smith [39] calculated $\det(S)_f$ on factor-closed sets [39, (5.)] and $\det[S]_f$ in a more special case [39, (3.)]. For example, Smith showed that

$$\det[(i, j)] = \phi(1)\phi(2)\cdots\phi(n),$$

where (i, j) is the GCD of i and j, and ϕ is Euler's totient function. Since Smith, a large number of papers on this topic has been published in the literature. For general accounts, see [1,16,17,22,36]. We assume that the reader is familiar with the modern terminology of GCD and LCM matrices; see, e.g. [4,5,18,22,23,25].

Let f be an arithmetical function such that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+$. We refer to the matrix $(S)_f/[S]_f$ whose ij entry is $f((x_i, x_j))/f([x_i, x_j])$ as the GCD reciprocal LCM

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matrix on S with respect to f. The matrix $(S)_f/[S]_f$ is thus the Hadamard quotient of the matrices $(S)_f$ and $[S]_f$. The LCM reciprocal GCD matrix on S with respect to f is defined analogously. Some results on eigenvalues, norms, determinants and inverses of certain special cases of GCD reciprocal LCM matrices have been published in the literature (see [12–14,19,24,28,30–33,41,43]) and some results on determinants and inverses of certain special cases of unitary analogs of GCD reciprocal LCM matrices also have been published in the literature (see [28,29,30,40]). Wintner [43] and Lindqvist and Seip [24] apply deep analytic tools to obtain results on eigenvalues of the GCD reciprocal LCM matrix $[(i,j)^\alpha/[i,j]^\alpha]$. Ilmonen et al. [19] note that lattice-theoretic methods reveal some properties of eigenvalues of certain general meet and join matrices, including meet reciprocal join matrices. Korkee [20] considers determinants and inverses of meet reciprocal join matrices in a lattice-theoretic level but he does not consider the unitary case. The purpose of this article is to investigate the determinant of certain unitary analogs of GCD reciprocal LCM matrices in detail. This theme may seem a very narrow field of research but it leads, however, to some interesting questions in elementary number theory.

A divisor $d \in \mathbb{Z}_+$ of $n \in \mathbb{Z}_+$ is said to be a unitary divisor of n and is denoted by $d \| n$ if $(d, n/d) = 1$. For example, the unitary divisors of 72 $(= 2^3 3^2)$ are $1, 8, 9, 72$. If $d \| n$, we also say that n is a unitary multiple of d. The greatest common unitary divisor (GCUD) of m and n exists for all $m, n \in \mathbb{Z}_+$ but, unfortunately, the least common unitary multiple (LCUM) of m and n does not always exist. For example, the LCUM of 2 and 4 does not exist. For details of GCUD and LCUM problematics, see Section 2.1.

In this article we examine GCUD reciprocal LCUM matrices. In Section 2 we concentrate on the difficulty of the non-existence of the LCUM. We go through three different ways to overcome this difficulty. The first way is the elementary number-theoretic formula presented in [10]. The second way is the lattice-theoretic approach adopted in [21]. The third way has a bearing on topology and lattice theory. It has not previously been presented in the literature.

In Section 3 we calculate the determinant of the three GCUD reciprocal LCUM matrices with respect to f arising from the three different ways to extend the concept of LCUM in Section 2. In each of these cases the expressions of the determinant coincide but the function f belongs to different classes of arithmetical functions. These classes are generated by the various ways to cope with the LCUM problematics and may be considered appropriate unitary analogs of the class of semimultiplicative functions. Unitary analogs of semimultiplicative functions have not hitherto been studied in the literature. In Section 4 we indicate that these unitary analogs are surprisingly related to rational arithmetical functions.

Our investigation shows that the way to overcome the difficulty of the non-existence of the LCUM does make a difference and provides some new connections between different areas in elementary number theory.

2. Least common unitary multiple

In this section we go into the LCUM in detail. We first point out the difficulty of the non-existence of the LCUM (Section 2.1) and after that we present three different ways to overcome this difficulty (Sections 2.2–2.4).

2.1. *Non-existence of the LCUM*

It is well known that the set $\mathbb{Z}_+$ of positive integers is a poset under the usual divisibility relation. It is likewise well known that the GCD and LCM operations serve as the meet and the join on this poset. Thus $\mathbb{Z}_+$ is a lattice under the usual divisibility relation, known as the divisor lattice. We recall that a divisor $d \in \mathbb{Z}_+$ of $n \in \mathbb{Z}_+$ is said to be a unitary divisor of n and is denoted by $\mathrm{d} \parallel n$ if $(d, n/d) = 1$. The unitary divisors of a prime power p^a are 1 and p^a. A general formula for the unitary divisors of $n = \prod_{p \in \mathbb{P}} p^{n(p)}$ can be written as

$$\prod_{p \in \mathbb{P}} p^{i_p},$$

where i_p runs over the (one or two) values 0 and $n(p)$ for all primes p. The number of the unitary divisors of n is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime divisors of n with $\omega(1) = 0$. The concept of a unitary divisor originates from Vaidyanathaswamy [42] and was further studied for example by Cohen [7]. We denote the GCUD of m and n as $(m, n)_{\oplus\oplus}$. The GCUD of m and n exists for all $m,\ n \in \mathbb{Z}_+$ and

$$(m, n)_{\oplus\oplus} = \prod_{p \in \mathbb{P}} p^{\rho(m(p), n(p))},$$

where $\rho(m(p),\ n(p)) = m(p)$ if $m(p) = n(p)$, and $\rho(m(p),\ n(p)) = 0$ if $m(p) \neq n(p)$. We denote the LCUM of m and n as $[m, n]_{\oplus\oplus}$. The LCUM of m and n exists if and only if $m(p) = n(p)$, $n(p) = 0$ or $m(p) = 0$ for each prime p. For example, the LCUM of 2 and 4 does not exist and the LCUM of 18 and 45 exists and is equal to 90. We exhibit three solutions to overcome the difficulty of the non-existence of the LCUM.

2.2. *Pseudo-LCUM*

Hansen and Swanson [10] overcame the difficulty of the non-existence of the LCUM by defining

$$[m, n]^* = \frac{mn}{(m, n)_{\oplus\oplus}}. \tag{2.1}$$

It is easy to see that $[m, n]^*$ exists for all $m,\ n \in \mathbb{Z}_+$ and $[m, n]^* = [m, n]_{\oplus\oplus}$ when $[m, n]_{\oplus\oplus}$ exists. Naturally, $[m, n]^* \neq [m, n]_{\oplus\oplus}$ when $[m, n]_{\oplus\oplus}$ does not exist. For example, $[2, 4]^* = 8$ but $[2, 4]_{\oplus\oplus}$ does not exist. It is not reasonable to say that 8 is the LCUM of 2 and 4. We say that $[m, n]^*$ in (2.1) is the *pseudo-LCUM* of m and n.

2.3. *∞-extended LCUM*

Korkee [21] developed a lattice-theoretic extension of $\mathbb{Z}_+$ so that the LCUM always exists in the extension and coincides with the LCUM in $\mathbb{Z}_+$ for the positive integers possessing the LCUM in $\mathbb{Z}_+$. In fact, it is easy to see that the unitary divisibility relation is a partial ordering on $\mathbb{Z}_+$. The GCUD operation serves as the meet on this poset. Thus $\mathbb{Z}_+$ is a meet semilattice under the unitary divisibility relation. Unfortunately, however, it is not a lattice, since the LCUM does not always exist.

Korkee [21] embedded the unitary divisor meet semilattice $(\mathbb{Z}_+, \|)$ in a lattice by adding an element, denoted as ∞, so that each $n \in \mathbb{Z}_+$ is a unitary divisor of ∞. Then $(\mathbb{Z}_+ \cup \{\infty\}, \|)$ is a lattice. We denote the join of m and n in the lattice $(\mathbb{Z}_+ \cup \{\infty\}, \|)$ by $[m, n]^*_\infty$ and we say that $[m, n]^*_\infty$ is the ∞-*extended LCUM* of m and n. Then for m, $n \in \mathbb{Z}_+ \cup \{\infty\}$, $a = [m, n]^*_\infty$ if and only if (1) $m \| a$, $n \| a$, and (2) $m \| b$, $n \| b \Rightarrow a \| b$. We note that

$$[m, n]^*_\infty = \begin{cases} [m, n]_{\oplus\oplus} & \text{if } [m, n]_{\oplus\oplus} \text{ exists}, \\ \infty & \text{otherwise.} \end{cases}$$

For example, $[18, 45]^*_\infty = 90$ and $[2, 4]^*_\infty = \infty$. We also note that this approach may be considered the one-point compactification [8] of the discrete topological space $\mathbb{Z}_+$.

2.4. p^∞-extended LCUM

In this subsection we propose a new way to embed the unitary divisor meet semilattice $(\mathbb{Z}_+, \|)$ in a lattice. We apply the one-point compactification with respect to each set of prime powers $\{p^a\colon\ a = 0, 1, 2, \ldots\}$ and then apply the fundamental theorem of arithmetic. This lattice may be considered a refinement of the lattice adopted by Korkee [21].

Let p be a prime and denote $U_p = \{p^a\colon\ a = 0, 1, 2, \ldots\}$. Let (U_p, τ) denote the corresponding discrete topological space. Let (U_p^*, τ^*) be the one-point compactification [8] of (U_p, τ). We denote the added point (often called the point of infinity) by p^∞. Thus $U_p^* = \{p^a\colon a = 0, 1, 2, \ldots \text{ or } a = \infty\} = \{1, p, p^2, \ldots, p^\infty\}$ and $\tau^* = \tau \cup \tau_{p^\infty}$, where τ_{p^∞} consists of the sets of the form $G \cup \{p^\infty\}$, where $G \in \tau$ and $U_p \backslash G$ is compact in U_p. We define the multiplication $\odot$ on U_p^* as follows:

$$p^a \odot p^b = \begin{cases} p^\infty & \text{if } a \neq b \text{ with } 0 < a, b \leq \infty, \text{ or } a = b = \infty, \\ p^{\max\{a,b\}} & \text{otherwise.} \end{cases}$$

Then U_p^* is a commutative compact topological semigroup of idempotents with identity under the multiplication $\odot$. We say that $p^a \in U_p^*$ is a unitary divisor of $p^b \in U_p^*$ (denoted as $p^a \| p^b$) if there exists $p^c \in U_p^*$ such that $p^b = p^a \odot p^c$. Then $1 \| p^a \| p^\infty$ for all $a = 0, 1, 2, \ldots, \infty$ and $p^a \nparallel p^b$ for all $0 < a, b < \infty$ with $a \neq b$. It is easy to see that $(U_p^*, \|)$ is a lattice and $p^a \vee p^b = p^a \odot p^b$ for all a, $b = 0, 1, 2, \ldots, \infty$. For example, $1 \vee p = p$, $p \vee p^2 = p^\infty$, $p^2 \vee p^2 = p^2$, $p^2 \vee p^\infty = p^\infty$ for all $p \in \mathbb{P}$. We now define $\mathbb{Z}_+^*$ as the topological product

$$\mathbb{Z}_+^* = \prod_{p \in \mathbb{P}} U_p^*.$$

By Tychonoff's theorem, $\mathbb{Z}_+^*$ is a commutative compact topological semigroup of idempotents with identity under the point-wise multiplication $\odot$. We can also consider $\mathbb{Z}_+^*$ as the direct product

$$\mathbb{Z}_+^* = \prod_{p \in \mathbb{P}} U_p^*$$

of the lattices U_p^*. Then $(2^{a_2}, 3^{a_3}, 5^{a_5}, \dots) \parallel (2^{b_2}, 3^{b_3}, 5^{b_5}, \dots)$ if and only if $p^{a_p} \parallel p^{b_p}$ for all $p \in \mathbb{P}$, where for each $p \in \mathbb{P}$, the relation $p^{a_p} \parallel p^{b_p}$ is as in U_p^*. Further,

$$(2^{a_2}, 3^{a_3}, 5^{a_5}, \dots) \wedge (2^{b_2}, 3^{b_3}, 5^{b_5}, \dots) = (2^{a_2} \wedge 2^{b_2}, 3^{a_3} \wedge 3^{b_3}, 5^{a_5} \wedge 5^{b_5}, \dots)$$

and

$$(2^{a_2}, 3^{a_3}, 5^{a_5}, \dots) \vee (2^{b_2}, 3^{b_3}, 5^{b_5}, \dots) = (2^{a_2} \vee 2^{b_2}, 3^{a_3} \vee 3^{b_3}, 5^{a_5} \vee 5^{b_5}, \dots),$$

where for each $p \in \mathbb{P}$, the operations $p^{a_p} \wedge p^{b_p}$ and $p^{a_p} \vee p^{b_p}$ are as in U_p^*. For example, we have

$$(2, 3^2, 5, 7^3, 1, 1, \dots) \wedge (1, 3^2, 5^2, 7^\infty, 1, 1, \dots) = (1, 3^2, 1, 7^3, 1, 1, \dots)$$

and

$$(2, 3^2, 5, 7^3, 1, 1, \dots) \vee (1, 3^2, 5^2, 7^\infty, 1, 1, \dots) = (2, 3^2, 5^\infty, 7^\infty, 1, 1, \dots).$$

The relation $\parallel$ on $\mathbb{Z}_+^*$ may be referred to as an extended unitary divisibility relation and the binary operations $\wedge$ and $\vee$ on $\mathbb{Z}_+^*$ may be referred to as extended GCUD and LCUM operations. The unitary divisor meet semilattice $(\mathbb{Z}_+, \parallel)$ is thus embedded in the lattice $(\mathbb{Z}_+^*, \parallel)$. As an illustration, the sublattice $U_2^* \times U_3^*$ of the lattice $\mathbb{Z}_+^* = \prod_{p \in \mathbb{P}} U_p^*$ is shown in Figure 1.

We use the term p^∞*-extended LCUM* for the join in $(\mathbb{Z}_+^*, \parallel)$ and denote it by $[m, n]_{p^\infty}^*$. That is,

$$[m, n]_{p^\infty}^* = m \vee n.$$

Note that in this notation p is not a fixed prime. It merely indicates that we extend the concept of LCUM as in this section, that is, we insert the point of infinity in each U_p, $p \in \mathbb{P}$.

Example 2.1 If $[m, n]_{\oplus\oplus}$ exists, then $[m, n]_{p^\infty}^* = [m, n]_\infty^* = [m, n]^* = [m, n]_{\oplus\oplus}$. On the other hand, for example $[2, 4]^* = 8$, $[2, 4]_\infty^* = \infty$, $[2, 4]_{p^\infty}^* = 2^\infty$ and $[2, 4]_{\oplus\oplus}$ does not exist.

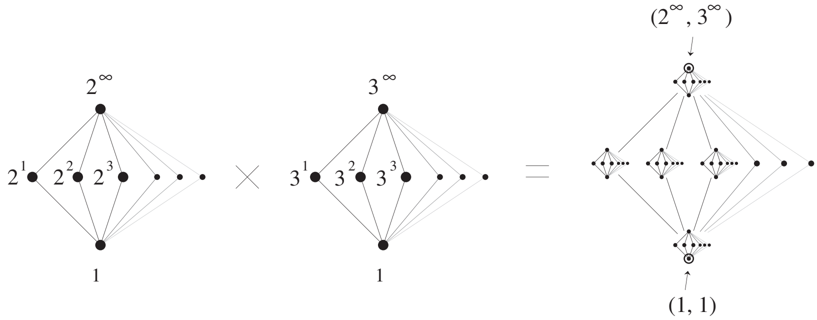

Figure 1. $U_2^* \times U_3^*$.

3. On determinants of GCUD reciprocal LCUM matrices

In this section we investigate the determinant of the three GCUD reciprocal LCUM matrices arising from the three solutions presented in Section 2 to the difficulty of the non-existence of the LCUM. In Section 3.1 we prove results from which in Section 3.2 we obtain our determinant formulas for GCUD reciprocal LCUM matrices as corollaries.

3.1. *General results*

Let $S=\{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers and let f be an arithmetical function such that $f(x)\neq 0$ for all $x\in\mathbb{Z}_+$. We first investigate the $n\times n$ matrix $(S^{\times})_f$ defined on the set S having

$$\frac{f^2((x_i, x_j)_{\oplus\oplus})}{f(x_i)(x_j)} \tag{3.1}$$

as its ij entry. Clearly such matrices are symmetric. Furthermore, rearrangements of the elements of S yield similar matrices and consequently we may always assume that $x_1 < x_2 < \cdots < x_n$. The unitary analog of the Möbius function is denoted by μ^* (see [7]). We adopt the notation

$$B_f^*(x_i) = \sum_{\substack{d\| x_i \\ d\not\| x_t \\ t<i}} g_f(d) \tag{3.2}$$

for all $i=1, 2, \ldots, n$, where

$$g_f(x) = \sum_{d\| x} f^2\left(\frac{x}{d}\right)\mu^*(d).$$

By the unitary analog of the Möbius inversion formula [7] we have

$$f^2(x) = \sum_{d\| x} g_f(d). \tag{3.3}$$

We now give a structure theorem for the matrix $(S^{\times})_f$ with respect to f on any set S.

THEOREM 3.1 *Let $S=\{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and let $\overline{S}=\{d_1, d_2, \ldots, d_m\}$ be the minimal GCUD-closed set containing S. Let f be an arithmetical function such that $f(x)\neq 0$ for all $x\in\mathbb{Z}_+$.*

Define the $n\times m$ matrix $C=(c_{ij})$ by

$$c_{ij} = \begin{cases} \dfrac{\sqrt{B_f^*(d_j)}}{f(x_i)} & \text{if } d_j \| x_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $(S^{\times})_f=CC^T$.

Proof We adopt an argument similar to that in Beslin and Ligh [3, Proposition 1].

The ij entry in CC^T is equal to

$$\sum_{k=1}^{m} c_{ik}c_{jk} = \sum_{\substack{d_k \| x_i \\ d_k \| x_j}} \frac{\sqrt{B_f^*(d_k)}}{f(x_i)} \frac{\sqrt{B_f^*(d_k)}}{f(x_j)} = \frac{1}{f(x_i)f(x_j)} \sum_{d_k \| (x_i,x_j)_{\oplus\oplus}} B_f^*(d_k). \tag{3.4}$$

It follows from (3.2) that

$$\sum_{d_k \| (x_i,x_j)_{\oplus\oplus}} B_f^*(d_k) = \sum_{d_k \| (x_i,x_j)_{\oplus\oplus}} \sum_{\substack{d \| d_k \\ d \nparallel d_t \\ d_t < d_k}} g_f(d) \tag{3.5}$$

and from (3.3) that

$$f^2((x_i, x_j)_{\oplus\oplus}) = \sum_{d \| (x_i,x_j)_{\oplus\oplus}} g_f(d). \tag{3.6}$$

We next show that the sums on the right-hand sides of (3.5) and (3.6) are equal. It is easy to see that these sums are non-repetitive, that is, each d is counted only once. Now, consider the sum on the right-hand side of (3.5). Let $d_k \| (x_i, x_j)_{\oplus\oplus}$ and $d \| d_k$. Then $d \| (x_i, x_j)_{\oplus\oplus}$. Thus every d occuring on the right-hand side of (3.5) occurs on the right-hand side of (3.6).

Conversely, consider the sum on the right-hand side of (3.6). Suppose that $d \| (x_i, x_j)_{\oplus\oplus}$, that is, $d \| x_i$, $d \| x_j$. Since $\overline{S}$ is GCUD-closed $(x_i, x_j)_{\oplus\oplus} = d_m$ for some m. Then $d \| d_m$. Let d_k be the least member of $\overline{S}$ such that $d \| d_k$. Then $d \nparallel d_t$ for $d_t < d_k$. We shall prove that $d_k \| (x_i, x_j)_{\oplus\oplus}$, that is, $d_k \| x_i$, $d_k \| x_j$. Firstly, we prove that $d_k \| x_i$. In fact, since $\overline{S}$ is GCUD-closed, $(x_i, d_k)_{\oplus\oplus} = d_r$ for some $d_r \le d_k$. Since $d \| x_i$, $d \| d_k$, we have $d \| d_r$. By the minimality of d_k we have $d_r = d_k$. Since $d_r \| x_i$, we obtain $d_k \| x_i$. Now the relation $d_k \| x_j$ follows by symmetry. Thus, $d_k \| (x_i, x_j)_{\oplus\oplus}$. This completes the proof that the sums on the right-hand sides of (3.5) and (3.6) are equal.

Now, by (3.4)–(3.6) we see that the ij entry in CC^T is equal to

$$\frac{1}{f(x_i)f(x_j)} \sum_{d_k \| (x_i,x_j)_{\oplus\oplus}} B_f^*(d_k) = \frac{1}{f(x_i)f(x_j)} \sum_{d \| (x_i,x_j)_{\oplus\oplus}} g_f(d) = \frac{f^2((x_i, x_j)_{\oplus\oplus})}{f(x_i)f(x_j)}.$$

Thus, on the basis of (3.1), we obtain $(S^\times)_f = CC^T$. ∎

Corollary 3.1 *Let* Λ *denote the* $m \times m$ *diagonal matrix*

$$\Lambda = \operatorname{diag}\big(B_f^*(d_1), B_f^*(d_2), \ldots, B_f^*(d_m)\big),$$

where $d_1, d_2, \ldots, d_m$ *are as defined in Theorem* 3.1, *and let* $H = (h_{ij})$ *denote the* $n \times m$ *matrix*

$$h_{ij} = \begin{cases} \dfrac{1}{f(x_i)} & \text{if } d_j \| x_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(S^{\times})_f = H\Lambda H^T.$$

Proof Corollary 3.1 follows directly from Theorem 3.1, noting that $C = H\Lambda^{1/2}$. ■

We next investigate the determinant of the matrix $(S^{\times})_f$ with respect to f on GCUD-closed set S.

THEOREM 3.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. Let f be an arithmetical function such that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+$. If S is GCUD-closed, then*

$$\det(S^{\times})_f = \prod_{k=1}^{n} \frac{B_f^*(x_k)}{f^2(x_k)}.$$

Proof Since S is a GCUD-closed set, we have $\overline{S} = S$. By Theorem 3.1, $(S^{\times})_f = CC^T$, where C is a lower triangular matrix with

$$c_{kk} = \frac{\sqrt{B_f^*(x_k)}}{f(x_k)}$$

for $k = 1, 2, \ldots, n$. Therefore,

$$\det(S^{\times})_f = \det(CC^T) = \det C \det C^T = (\det C)^2 = \prod_{k=1}^{n} \frac{B_f^*(x_k)}{f^2(x_k)}.$$ ■

We now investigate the determinant of the matrix $(S^{\times})_f$ with respect to f on an arbitrary finite set S.

THEOREM 3.3 *Let $\overline{S} = \{x_1, x_2, \ldots, x_n, \ldots, x_{n+s}\}$ be the minimal GCUD-closed set containing $S = \{x_1, x_2, \ldots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$ and $x_{n+1} < x_{n+2} < \cdots < x_{n+s}$. Let f be an arithmetical function such that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+$. Then*

$$\det(S^{\times})_f = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le n+s} \left(\det H_{(k_1, k_2, \ldots, k_n)}\right)^2 B_f^*(x_{k_1}) B_f^*(x_{k_2}) \cdots B_f^*(x_{k_n}),$$

where $H_{(k_1, k_2, \ldots, k_n)}$ is the submatrix of H consisting of its k_1-th, k_2-th, $\ldots$, k_n-th columns and H is as given in Corollary 3.1.

Proof Theorem 3.1 says that $(S^{\times})_f = CC^T$ and thus $\det(S^{\times})_f = \det(CC^T)$. By the Cauchy–Binet formula [9, p. 9] we obtain

$$\det(CC^T) = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le n+s} \det C_{(k_1, k_2, \ldots, k_n)} \det C^T_{(k_1, k_2, \ldots, k_n)},$$

where $C_{(k_1, k_2, \ldots, k_n)}$ is the submatrix of C consisting of its k_1-th, k_2-th, $\ldots$, k_n-th columns. Further

$$\det C^T_{(k_1, k_2, \ldots, k_n)} = \det C_{(k_1, k_2, \ldots, k_n)} = \sqrt{B_f^*(x_{k_1}) B_f^*(x_{k_2}) \cdots B_f^*(x_{k_n})} \det H_{(k_1, k_2, \ldots, k_n)}$$

and hence

$$\det(S^{\times})_f = \sum_{1\le k_1<k_2<\cdots<k_n\le n+s} \left(\det H_{(k_1,k_2,\ldots,k_n)}\right)^2 B_f^*(x_{k_1})B_f^*(x_{k_2})\cdots B_f^*(x_{k_n}). \qquad \blacksquare$$

3.2. *GCUD reciprocal LCUM matrices*

We now define the GCUD reciprocal LCUM matrices arising from the extensions of the concept of LCUM presented in Section 2.

Definition 3.1 Let $S=\{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and let f be an arithmetical function such that $f(x)\neq 0$ for all $x\in\mathbb{Z}_+$. We denote by $(S^{\oplus\oplus})_f$ the $n\times n$ matrix having $f((x_i, x_j)_{\oplus\oplus})$ as its ij entry, and similarly we denote by $[S^*]_f$ the $n\times n$ matrix having $f([x_i, x_j]^*)$ as its ij entry. Now, we denote by $(S^{\oplus\oplus})_f/[S^*]_f$ the $n\times n$ matrix having $f((x_i, x_j)_{\oplus\oplus})/f([x_i, x_j]^*)$ as its ij entry, that is, $(S^{\oplus\oplus})_f/[S^*]_f$ is the Hadamard quotient of the matrices $(S^{\oplus\oplus})_f$ and $[S^*]_f$. The matrix $(S^{\oplus\oplus})_f/[S^*]_f$ may be referred to as the *GCUD reciprocal pseudo-LCUM matrix* on S with respect to f. The matrices $[S^*_\infty]_f$, $[S^*_{p\infty}]_f$, $(S^{\oplus\oplus})_f/[S^*_\infty]_f$ and $(S^{\oplus\oplus})_f/[S^*_{p\infty}]_f$ are defined analogously.

We next convert Theorems 3.2 and 3.3 to the various GCUD reciprocal LCUM matrices. This is possible for certain types of functions f. We introduce these classes of functions in the next definition.

Definition 3.2 Let $T\subseteq\mathbb{Z}_+$. We say that a complex-valued function f defined on the set $\mathbb{Z}_+$ (that is, an arithmetical function f) is *pseudo-unitarily semimultiplicative* on T if

$$f((m,n)_{\oplus\oplus})f([m,n]^*) = f(m)f(n),\ \forall m,\ n\ \in\ T, \tag{3.7}$$

a complex-valued function g defined on the set $\mathbb{Z}_+\cup\{\infty\}$ is ∞-*unitarily semimultiplicative* on T if

$$g((m,n)_{\oplus\oplus})g([m,n]^*_\infty) = g(m)g(n),\ \forall m,n\in T \tag{3.8}$$

and a complex-valued function h defined on the set $\mathbb{Z}^*_+$ is p^∞-*unitarily semimultiplicative* on T if

$$h((m,n)_{\oplus\oplus})h([m,n]^*_{p\infty}) = h(m)h(n),\ \forall m,n\in T. \tag{3.9}$$

Remark 3.1 An arithmetical function f is said to be semimultiplicative [38, p. 237] if

$$f((m,n))f([m,n]) = f(m)f(n),\ \forall m,n\in\mathbb{Z}_+. \tag{3.10}$$

The functions in Definition 3.2 may thus be referred to as unitary analogs of semimultiplicative functions on T. Note that these unitary analogs depend on the way to define the concept of LCUM. We analyse these classes of functions of Definition 3.2 in Section 4. For simplicity we there assume that $T=\mathbb{Z}_+$.

The following three corollaries are direct consequences of Theorem 3.2.

COROLLARY 3.2 *Let* $S=\{x_1,x_2,\dots,x_n\}$ *be a set of distinct positive integers and let* f *be a complex-valued function defined on the set* $\mathbb{Z}_+$ *such that* $f(x)\neq 0$ *for all* $x\in\mathbb{Z}_+$*. If* S *is GCUD-closed and* f *is pseudo-unitarily semimultiplicative on* S*, then*

$$\det\big[(S^{\oplus\oplus})_f/[S^*]_f\big]=\prod_{k=1}^{n}\frac{B_f^*(x_k)}{f^2(x_k)}.$$

COROLLARY 3.3 *Let* $S=\{x_1,x_2,\dots,x_n\}$ *be a set of distinct positive integers and let* f *be a complex-valued function defined on the set* $\mathbb{Z}_+\cup\{\infty\}$ *such that* $f(x)\neq 0$ *for all* $x\in\mathbb{Z}_+\cup\{\infty\}$*. If* S *is GCUD-closed and* f *is* ∞*-unitarily semimultiplicative on* S*, then*

$$\det\big[(S^{\oplus\oplus})_f/[S^*_\infty]_f\big]=\prod_{k=1}^{n}\frac{B_f^*(x_k)}{f^2(x_k)}.$$

COROLLARY 3.4 *Let* $S=\{x_1,x_2,\dots,x_n\}$ *be a set of distinct positive integers and let* f *be a complex-valued function defined on the set* $\mathbb{Z}_+^*$ *such that* $f(x)\neq 0$ *for all* $x\in\mathbb{Z}_+^*$*. If* S *is GCUD-closed and* f *is* p^∞*-unitarily semimultiplicative on* S*, then*

$$\det\Big[(S^{\oplus\oplus})_f/[S^*_{p^\infty}]_f\Big]=\prod_{k=1}^{n}\frac{B_f^*(x_k)}{f^2(x_k)}.$$

Example 3.1 Let $S=\{x_1,x_2,\dots,x_n\}$ be a GCUD-closed ordered set of distinct positive integers and let α be a real number. Let $f(x)=x^\alpha$ for all $x\in\mathbb{Z}_+$. Then

$$\det\big[(S^{\oplus\oplus})_f/[S^*]_f\big]=\prod_{k=1}^{n}\frac{B_\alpha^*(x_k)}{(x_k)^{2\alpha}},$$

where

$$B_\alpha^*(x_i)=\sum_{\substack{d\|x_i\\ d\not\| x_t\\ t<i}}J_{2\alpha}^*(d)$$

for all $i=1,2,\dots,n$, and $J_{2\alpha}^*$ is the unitary analogue of the Jordan totient function. The function J_u^*, $u\in\mathbb{R}$, is defined as $J_u^*(n)=\sum_{d\|n}d^u\mu^*(n/d)$ and was first introduced for $u\in\mathbb{Z}_+$ in [27].

The following three corollaries are direct consequences of Theorem 3.3. In these corollaries $\overline{S}=\{x_1,x_2,\dots,x_n,\dots,x_{n+s}\}$ is the minimal GCUD-closed set containing $S=\{x_1,x_2,\dots,x_n\}$, where $x_1<x_2<\cdots<x_n$ and $x_{n+1}<x_{n+2}<\cdots<x_{n+s}$.

COROLLARY 3.5 *Let* f *be a complex-valued function defined on the set* $\mathbb{Z}_+$ *such that* $f(x)\neq 0$ *for all* $x\in\mathbb{Z}_+$*. If* f *is pseudo-unitarily semimultiplicative on* S*, then*

$$\det\big[(S^{\oplus\oplus})_f/[S^*]_f\big]=\sum_{1\le k_1<k_2<\cdots<k_n\le n+s}\big(\det H_{(k_1,k_2,\dots,k_n)}\big)^2B_f^*(x_{k_1})B_f^*(x_{k_2})\cdots B_f^*(x_{k_n}),$$

where $H_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of H consisting of its k_1-th, k_2-th, …, k_n-th columns.

COROLLARY 3.6 *Let f be a complex-valued function defined on the set $\mathbb{Z}_+ \cup \{\infty\}$ such that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+ \cup \{\infty\}$. If f is ∞-unitarily semimultiplicative on S, then*

$$\det\left[(S^{\oplus\oplus})_f/[S^*_\infty]_f\right] = \sum_{1\le k_1<k_2<\cdots<k_n\le n+s} \left(\det H_{(k_1,k_2,\ldots,k_n)}\right)^2 B^*_f(x_{k_1})B^*_f(x_{k_2})\cdots B^*_f(x_{k_n}).$$

COROLLARY 3.7 *Let f be a complex-valued function defined on the set $\mathbb{Z}^*_+$ such that $f(x) \neq 0$ for all $x \in \mathbb{Z}^*_+$. If f is p^∞-unitarily semimultiplicative on S, then*

$$\det\left[(S^{\oplus\oplus})_f/[S^*_{p^\infty}]_f\right] = \sum_{1\le k_1<k_2<\cdots<k_n\le n+s} \left(\det H_{(k_1,k_2,\ldots,k_n)}\right)^2 B^*_f(x_{k_1})B^*_f(x_{k_2})\cdots B^*_f(x_{k_n}).$$

Remark 3.2 We considered above determinants of GCUD reciprocal LCUM matrices with respect to f. Taking $f = 1/h$ we obtain expressions for determinants of LCUM reciprocal GCUD matrices with respect to h. Thus in this case we know a relation between the determinant of a matrix and its Hadamard inverse.

Remark 3.3 The semi-unitary LCM of m and n is the least unitary multiple of m which is a multiple of n, and the semi-unitary GCD of m and n is the greatest unitary divisor of m which is a divisor of n. It would be possible to examine the determinants of semi-unitary LCM reciprocal semi-unitary GCD matrices with the same methods as above. We do not go into the details here.

4. Properties of the unitary analogs of semimultiplicative functions

In Definition 3.2 we introduced certain classes of functions (depending on the way to define the LCUM) for the purpose of our formulas for determinants. These classes of functions have not previously been studied in the literature. This suggests we investigate their basic properties. The classes of functions to be studied are pseudo-unitarily semimultiplicative functions, ∞-unitarily semimultiplicative functions and p^∞-unitarily semimultiplicative functions on T. For the sake of simplicity we confine ourselves to the case $T = \mathbb{Z}_+$. It appears that there are differences between these classes of functions and thus this shows that the way to overcome the difficulty of the non-existence of the LCUM does make a difference. At first we review some preliminaries on arithmetical functions.

4.1. *Preliminaries on arithmetical functions*

An arithmetical function f is said to be quasimultiplicative if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \tag{4.1}$$

for all $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$. Quasimultiplicative functions f may also be characterized as semimultiplicative functions with $f(1) \neq 0$, see (3.10). A quasimultiplicative function f is said to be multiplicative if $f(1) = 1$. An arithmetical function f with $f(1) \neq 0$ is multiplicative if and only if $f/f(1)$ is quasimultiplicative.

A multiplicative function is totally determined by its values at prime powers. A multiplicative function f is said to be completely multiplicative if

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbb{Z}_+$. A completely multiplicative function is totally determined by its values at primes. A multiplicative function f is said to be strongly multiplicative if $f(p^a) = f(p)$ for all prime powers p^a with $a \geq 1$.

The Dirichlet convolution of arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, serves as the identity under the Dirichlet convolution. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$ and it is denoted as f^{-1}.

An arithmetical function f is said to be a rational arithmetical function of order (r, s) if

$$f = g_1 * g_2 * \cdots * g_r * h_1^{-1} * h_2^{-1} * \cdots * h_s^{-1},$$

where $g_1, g_2, \ldots, g_r, h_1, h_2, \ldots, h_s$ are completely multiplicative functions. Rational arithmetical functions of order $(1, 1)$ are called totients.

The Bell series of a multiplicative function f to the base p is defined as the formal power series

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n.$$

It is known that a multiplicative function f is a rational arithmetical function of order (r, s) if and only if for each prime p, $f_p(x)$ is of the form

$$f_p(x) = \frac{1 + a_1(p)x + a_2(p)x^2 + \cdots + a_s(p)x^s}{1 + b_1(p)x + b_2(p)x^2 + \cdots + b_r(p)x^r}, \tag{4.2}$$

where $a_1(p), a_2(p), \ldots, a_s(p), b_1(p), b_2(p), \ldots, b_r(p)$ are complex numbers.

General material on arithmetical functions can be found from the books [2,26,36,37,38]. Reference to rational arithmetical functions is made in [6,11,15,35,42].

4.2. *Pseudo-unitarily semimultiplicative functions*

In this section we examine properties of pseudo-unitarily semimultiplicative functions on $\mathbb{Z}_+$.

THEOREM 4.1 *If f is pseudo-unitarily semimultiplicative on $\mathbb{Z}_+$ and $f(1) = 1$, then f is multiplicative and*

$$f(p^a) = f(p)^{a-2} f(p^2) \tag{4.3}$$

for all primes p and integers $a \geq 2$.

Proof Taking $m, n \in \mathbb{Z}_+$ with $(m,n)=1$ in (3.7) we obtain $f(mn)=f(m)f(n)$. This means that f is multiplicative.

We proceed by induction on a to prove that (4.3) holds. For $a=2$, Equation (4.3) holds trivially. Assume then that (4.3) holds for $a=k$ ($k \geq 2$). We prove that (4.3) holds for $a=k+1$. From the assumption $f(1)=1$ and the properties of GCUD and pseudo-LCUM we obtain

$$f(p^{k+1}) = f(1)f(p^{k+1}) = f((p,p^k)_{\oplus\oplus})f([p,p^k]^*).$$

Applying the above equation, Equation (3.7) and the induction assumption we obtain

$$f(p^{k+1}) = f(p)f(p^k) = f(p)^{k-1}f(p^2).$$

This completes the induction. ■

Remark 4.1 The converse of Theorem 4.1 does not hold.

THEOREM 4.2 *If f is multiplicative and satisfies* (4.3) *for all primes p and integers $a \geq 2$, then f is a rational arithmetical function of order* $(1,2)$.

Proof It is sufficient to show that the Bell series of f to the base p is of the form (4.2) with $r=1$ and $s=2$. On the basis of (4.3) we have

$$\begin{aligned} f_p(x) &= 1 + f(p)x + \sum_{n=2}^{\infty} f(p)^{n-2} f(p^2)x^n \\ &= 1 + f(p)x + f(p^2)x^2 \sum_{n=0}^{\infty} f(p)^n x^n. \end{aligned}$$

According to the summation of a geometric series we obtain

$$\begin{aligned} f_p(x) &= 1 + f(p)x + \frac{f(p^2)x^2}{1 - f(p)x} \\ &= \frac{1 - (f(p)^2 - f(p^2))x^2}{1 - f(p)x}. \end{aligned}$$

This completes the proof. ■

Remark 4.2 The converse of Theorem 4.2 does not hold.

Remark 4.3 Replacing the condition $f(1)=1$ with the condition $f(1) \neq 0$ would lead to quasimultiplicative functions in Theorems 4.1 and 4.2. We do not present the details.

Our investigation above showed that the class of pseudo-unitarily semimultiplicative functions on $\mathbb{Z}_+$ having $f(1)=1$ is a subclass of the class of functions satisfying $f(p^a)=f(p)^{a-2}f(p^2)$ for all $a \geq 2$ and for all primes p. Further, the latter class is a subclass of the class of rational arithmetical functions of order $(1,2)$. It is clear that the class of rational arithmetical functions of order $(1,2)$ is a subclass of the class of multiplicative functions. Note that if f is pseudo-unitarily semimultiplicative on $\mathbb{Z}_+$, then f is pseudo-unitarily semimultiplicative on T for all $T \subseteq \mathbb{Z}_+$.

4.3. ∞-unitarily semimultiplicative functions

We now examine properties of ∞-unitarily semimultiplicative functions on $\mathbb{Z}_+$.

THEOREM 4.3 *If $f:\mathbb{Z}_+\cup\{\infty\}\to\mathbb{C}$ is ∞-unitarily semimultiplicative on $\mathbb{Z}_+$ such that $f(p^a)\neq 0$ for all primes p and positive integers a, then f is a constant function.*

Proof Applying (3.8) with $(m,n)=1$ we see that (4.1) holds for all $m, n\in\mathbb{Z}_+$ with $(m,n)=1$, that is, f is quasimultiplicative on $\mathbb{Z}_+$. Now, let p be a prime and let a and b be positive integers with $a\neq b$. Then $[p^a,p^b]^*_\infty=\infty$, $(p^a,p^b)_{\oplus\oplus}=1$. Taking $m=p^a$ and $n=p^b$ in (3.8) we obtain

$$f(p^a)f(p^b)=f((p^a,p^b)_{\oplus\oplus})f([p^a,p^b]^*_\infty)=f(1)f(\infty)=c \text{ (a constant).} \tag{4.4}$$

Applying (4.4) to $a=1$, $b=u(u\geq 2)$ and $a=1$, $b=2$ we see that $f(p)f(p^u)=f(p)f(p^2)=c$, and since $f(p)\neq 0$, we obtain $f(p^2)=f(p^u)$ for all $u\geq 2$. Similarly, applying (4.4) to $a=2$, $b=u(u\geq 3)$ and $a=2$, $b=1$ and noting that $f(p^2)\neq 0$, we obtain $f(p^u)=f(p)$ for all $u\geq 3$. Therefore,

$$f(p^u)=f(p) \text{ for all } u\geq 1. \tag{4.5}$$

Now let p_1 and p_2 be distinct primes and take $m=p_1p_2$ and $n=p_1p_2^2$ in (3.8). Then we obtain

$$f(p_1)f(\infty)=f(p_1p_2)f(p_1p_2^2).$$

Applying (4.1) with $m=p_1, n=p_2$ we arrive at

$$f(p_1)f(\infty)=f(p_1)f(p_2)f(p_1)f(p_2^2)f(1)^{-2}.$$

Then, on the basis of (4.4) we deduce that

$$f(p_1)=f(1)=f(\infty). \tag{4.6}$$

From (4.5) and (4.6) we see that $f(p^u)=f(\infty)$ for all primes p and positive integers u. Finally, using (4.1) we can verify that

$$f(N)=f\left(\prod_{i=1}^r p_i^{N_i}\right)=f(1)^{-r+1}\prod_{i=1}^r f(p_i^{N_i})=f(1)=f(\infty) \text{ (a constant)}$$

for all $N\geq 2$ and therefore f is a constant function. ■

Remark 4.4 Clearly if $f:\mathbb{Z}_+\cup\{\infty\}\to\mathbb{C}$ is a constant function, then f is ∞-unitarily semimultiplicative on $\mathbb{Z}_+$.

Remark 4.5 In this article we do not analyse ∞-unitarily semimultiplicative functions with $f(p^a)=0$ for a positive integer a, since in our determinant formulas in Section 3 we assume that $f(x)\neq 0$ for all $x\in\mathbb{Z}_+\cup\{\infty\}$.

Remark 4.6 If f is a constant function, then the determinant in Corollaries 3.3 and 3.6 is zero. However, these corollaries are not trivial, since there exists various sets S for which there exists non-constant ∞-unitarily semimultiplicative functions on S and the determinant is also non-zero.

4.4. p^∞-unitarily semimultiplicative functions

We next examine properties of p^∞-unitarily semimultiplicative functions on $\mathbb{Z}_+$.

THEOREM 4.4 *If $f:\mathbb{Z}_+^* \to \mathbb{C}$ is p^∞-unitarily semimultiplicative on $\mathbb{Z}_+$ such that $f(1)=1$ and $f(p^a)\neq 0$ for all primes p and positive integers a, then*

(1) *the restriction $f_{\mathbb{Z}}$ of f to $\mathbb{Z}$ is a strongly multiplicative function (that is, $f(p^a)=f(p)$)*
(2) *$f(p^\infty)=f(p)^2$ for all primes p.*

Proof Taking $m,\ n\in\mathbb{Z}_+$ with $(m,n)=1$ in (3.9) we see that $f_{\mathbb{Z}}$ is multiplicative. Now, let p be a prime. Then for all positive integers a and b with $a\neq b$, taking $m=p^a$ and $n=p^b$ in (3.9), we obtain

$$f(p^a)f(p^b)=f(p^\infty).$$

From this we can deduce after some calculations that $f(p^a)=f(p)$ for all positive integers a and $f(p^\infty)=f(p)^2$. This completes the proof. ∎

Remark 4.7 The converse of Theorem 4.4 does not hold. However, the following equivalence holds.

Let $f:\mathbb{Z}_+^*\to\mathbb{C}$ be a function such that $f(1)=1$ and $f(p^a)\neq 0$ for all primes p and positive integers a. Then

$$f((m,n)_{\oplus\oplus})f([m,n]^*_{p^\infty})=f(m)f(n)\ \forall m,n\in\mathbb{Z}_+^*$$

if and only if

$$\begin{aligned} f(mn)&=f(m)f(n) \text{ for all } m,n\in\mathbb{Z}_+^* \text{with } (m,n)=1,\\ f(p^a)&=f(p) \text{ for all primes } p \text{ and } a\in\mathbb{Z}_+,\\ f(p^\infty)&=f(p)^2 \text{ for all primes } p.\end{aligned}$$

Here, for $m,n\in\mathbb{Z}_+^*, (m,n)_{\oplus\oplus}=m\wedge n, [m,n]^*_{p^\infty}=m\vee n$ and (m,n) is defined as the sequence $(2^{\min\{m(2),n(2)\}},\ 3^{\min\{m(3),n(3)\}},\dots,p^{\min\{m(p),n(p)\}},\dots)$ in $\mathbb{Z}_+^*=\prod_{p\in\mathbb{P}}U_p^*$ or equivalently (m,n) is defined as the formal product

$$(m,n)=\prod_{p\in\mathbb{P}}p^{\min\{m(p),n(p)\}},$$

where $\min\{a,\infty\}=a$ for all $a\in\mathbb{Z}_+\cup\{\infty\}$ (Section 2.4.). We do not present the proof.

Example 4.1 Let $f:\mathbb{Z}_+^*\to\mathbb{C}$ be defined as

$$\begin{aligned} f(1)&=1,\\ f(p^a)&=p \text{ for all primes } p \text{ and } a\in\mathbb{Z}_+,\\ f(p^\infty)&=p^2 \text{ for all primes } p,\\ f(mn)&=f(m)f(n) \text{ for all } m,n\in\mathbb{Z}_+^* \text{ with } (m,n)=1.\end{aligned}$$

Now f is p^∞-unitarily semimultiplicative function on $\mathbb{Z}_+$, but f is not ∞-unitarily semimultiplicative function on $\mathbb{Z}_+$.

THEOREM 4.5 *If* $f:\mathbb{Z}_+\to\mathbb{C}$ *is strongly multiplicative, then* f *is a rational arithmetical function of order* $(1, 1)$ (*that is,* f *is a totient*).

Proof If f is strongly multiplicative, then direct calculation shows that

$$f_p(x) = 1 + f(p)\sum_{n=1}^{\infty} x^n = 1 + \frac{f(p)x}{1-x}$$
$$= \frac{1-(1-f(p))x}{1-x}.$$

Thus, according to (4.2), f is a rational arithmetical function of order $(1, 1)$.
This completes the proof. ■

Remark 4.8 The converse of Theorem 4.5 does not hold.

We have shown that pseudo-unitarily semimultiplicative functions, ∞-unitarily semimultiplicative functions and p^{∞}-unitarily semimultiplicative functions form classes of functions that differ from each other. Thus we have shown that it does make a difference how we decide to overcome difficulties occuring when defining the LCUM of positive integers.

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