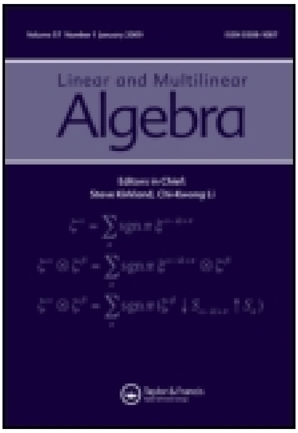

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A topological approach to divisibility of arithmetical functions and GCD matrices

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A topological approach to divisibility of arithmetical functions and GCD matrices

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Considering lower closed sets as closed sets on a preposet $(P, \leq)$, we obtain an Alexandroff topology on P. Then order preserving functions are continuous functions. In this article we investigate order preserving properties (and thus continuity properties) of integer-valued arithmetical functions under the usual divisibility relation of integers and power GCD matrices under the divisibility relation of integer matrices.

Keywords: preposet; topology; arithmetical function; GCD matrix; divisor; order preserving function; continuous function; Smith normal form

AMS Subject Classifications: 11A25; 11C20; 15B36; 06A06; 54A05

1. Introduction

A set P equipped with a reflexive and transitive relation $\leq$ is said to be a preordered set, a quasiordered set or a preposet. If $\leq$ is also antisymmetric, $(P, \leq)$ is referred to as a partially ordered set or a poset. Let $(P, \leq)$ be a preposet. A set $X \subseteq P$ is said to be lower closed if $y \in X$ for each $y \leq x \in X$. A lower closed set is also referred to as a lower set or a down set. Considering lower closed sets as closed sets we obtain a topology on P. This topology is an Alexandroff topology. It can be easily seen that order preserving functions from a preposet to a preposet are continuous functions with respect to the induced topologies. Cf. [13,49]. In Section 2 we present the details.

It is easy to see that $(\mathbb{Z}^+, |)$ is a poset and even a lattice. It is likewise easy to see that $(\mathbb{Z}, |)$ is a preposet, where $a \mid b$ and $b \mid a$ if and only if $a = \pm b$. Thus the divisibility relation induces topologies on $\mathbb{Z}^+$ and $\mathbb{Z}$. It is well known [4] that Euler's totient function ϕ possesses the divisibility property

$$d \mid n \Rightarrow \phi(d) \mid \phi(n). \tag{1}$$

This means that ϕ is a continuous function with respect to the induced topologies. This kind of approach has already been adopted in [42,44]. In Section 3 we further analyse (1).

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers. The GCD matrix (S) on S is defined as the $n \times n$ matrix whose ij entry is the greatest common divisor

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of x_i and x_j. The LCM matrix $[S]$ on S is defined analogously. The study of GCD and LCM matrices was begun in 1876 by Smith [48], who showed among others that

$$\det((i,j)) = \phi(1)\phi(2)\cdots\phi(n), \tag{2}$$

where (i,j) is the greatest common divisor of i and j, and ϕ is Euler's totient function. Since Smith, a large number of papers on this topic has been published in the literature. For general accounts, see [2,8,20,23,36,45].

Let $M_n(\mathbb{Z})$ denote the ring of $n \times n$ matrices over the integers. We say that $A \in M_n(\mathbb{Z})$ divides $B \in M_n(\mathbb{Z})$ from the left if there exists $C \in M_n(\mathbb{Z})$ such that $B = AC$ [34,39,54]. Divisibility from the right is defined analogously. It is possible that $A \in M_n(\mathbb{Z})$ divides $B \in M_n(\mathbb{Z})$ from the left but $A \in M_n(\mathbb{Z})$ does not divide $B \in M_n(\mathbb{Z})$ from the right (and vice versa). Since GCD and LCM matrices are symmetric, divisibilities from the left and right coincide.

Matrix divisibility from the left (right) is a preorder on $M_n(\mathbb{Z})$. Thus it induces a topology on $M_n(\mathbb{Z})$. In Section 4 we present certain matrix divisibility properties of the so-called power GCD and LCM matrices in terms of continuity with respect to appropriate topologies. In Section 5 we present a result on the Smith normal form of GCD matrices in terms of continuity with respect to appropriate topologies. Topological aspects have not hitherto been presented in the context of divisibility of integer matrices.

2. Topology on preposets

Let $(P, \leq)$ be a preposet. For each $X \subseteq P$, let $\overline{X}$ denote the intersection of all lower closed sets containing X. Then $\overline{X}$ is lower closed. It is easy to see that $\overline{X} = \cup_{x \in X} I_x$, where $I_x = \{y \leq x \mid y \in P\}$ is the principal ideal generated by x.

The mapping $f: \mathcal{P}(P) \to \mathcal{P}(P)$, $f(X) = \overline{X}$, is a closure operator, since $\overline{\emptyset} = \emptyset$, $X \subseteq \overline{X}$, $\overline{\overline{X}} = \overline{X}$ and $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ for all X, $Y \in \mathcal{P}(P)$ (here $\mathcal{P}(P)$ is the power set of P).

This suggests that we define a topology on P. Namely, $X \subseteq P$ is closed if $X = \overline{X}$, and $X \subseteq P$ is open if the complement of X is closed. We denote this topology as $\mathcal{L}$. For basics of topology we refer to [10,11].

The topology $\mathcal{L}$ is a T_0-topology if and only if $(P, \leq)$ is a poset [49]. To prove this, assume first that $(P, \leq)$ is a poset. Then for each x, $y \in P$ with $x \neq y$ we have $\overline{\{x\}} \neq \overline{\{y\}}$ and thus $\mathcal{L}$ is a T_0-topology. Assume then that $\mathcal{L}$ is a T_0-topology and show that $\leq$ is antisymmetric. Suppose, on the contrary, that there exist x, $y \in P$ such that $x \leq y$, $y \leq x$ and $x \neq y$. Since $\mathcal{L}$ is a T_0-topology, there is an open set U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Without loss of generality, we may assume that $x \in U$ and $y \notin U$. Then $P \backslash U$ is lower closed such that $x \notin P \backslash U$ and $y \in P \backslash U$. Thus $x \leq z$ for all $z \in P \backslash U$ and since $y \in P \backslash U$, we have $x \nleq y$, a contradiction. This shows that $\leq$ is antisymmetric.

A poset is said to be an antichain if any two elements are incomparable. It is easy to see that the topology $\mathcal{L}$ is a T_1-topology if and only if $(P, \leq)$ is an antichain. In fact, then and only then each $\{x\}$ is a lower closed subset of P.

Let $(P, \leq)$ be a preposet with least element(s). Then P with $\mathcal{L}$ is a compact, connected topological space. This follows from the observation that each closed set contains the least element(s) and thus P is the only open set containing the least element(s). Let $(P, \leq)$ be a poset with the least element 0. Then $x \in P$ is an atom if and only if $\overline{\{x\}}$ has exactly two elements.

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two preposets. A function $f : P_1 \to P_2$ is said to be order preserving or monotone if

$$x \leq_1 y \Rightarrow f(x) \leq_2 f(y) \tag{3}$$

for all $x, y \in P_1$. Let S_1 and S_2 be two topological spaces. A function $f: S_1 \to S_2$ is said to be continuous if the inverse image of each open set in S_2 is open in S_1. Equivalently, $f: S_1 \to S_2$ is continuous if and only if the inverse image of each closed set in S_2 is closed in S_1.

THEOREM 2.1 *A function f: $P_1 \to P_2$ is order preserving with respect to preorders $\leq_1$ and $\leq_2$ if and only if f is continuous with respect to topologies $\mathcal{L}_1$ and $\mathcal{L}_2$* [49].

Proof Assume that f is continuous and show that it is order preserving. Let $x \leq_1 y$. Then $x \in \overline{\{y\}}$ and thus $f(x) \in f\big(\overline{\{y\}}\big)$. Since f is continuous, $f\big(\overline{\{y\}}\big) \subseteq \overline{\{f(y)\}}$, and therefore $f(x) \in \overline{\{f(y)\}}$. This means that $f(x) \leq f(y)$.

Assume then that f is order preserving. We show that $f\big(\overline{X}\big) \subseteq \overline{f(X)}$ for all nonempty subset X of P_1. Let $z \in f\big(\overline{X}\big)$. Then there exist $y \in X$ and $x \leq y$ such that $z = f(x)$, and since f is order preserving, we have $z = f(x) \leq f(y) \in f\big(\overline{X}\big)$. Therefore, $z \in \overline{f(X)}$. This shows that $f\big(\overline{X}\big) \subseteq \overline{f(X)}$. ■

Remark An Alexandrov space is a topological space in which the intersection of any family of open sets is open [1,5,38]. Kukieła [38] generates Alexandrov spaces from preposets so that lower closed sets are open sets. In this article we generate Alexandrov spaces from preposets so that lower closed sets are closed sets and upper closed sets are open sets. The results in Section 2 of this article naturally go through dually when upper closed sets are closed sets and lower closed sets are open sets. The approach adopted in this article is similar to that of Porubský and Rizza [42,44] and is better in the sense that positive integers under the usual and unitary divisibility and integer matrices under the left and right divisibility induce *compact* topological spaces as will be seen in Sections 3 and 4.

3. Arithmetical functions

An arithmetical function f is said to be multiplicative if

$$f(mn) = f(m)f(n) \tag{4}$$

whenever $(m, n) = 1$. A multiplicative function f is said to be completely multiplicative if (4) holds for all m and n. A divisor d of n is said to be a unitary divisor (written as $d \| n$) if $(d, n/d) = 1$. It is easy to see that an arithmetical function f is multiplicative if and only if

$$d \| n \Rightarrow f(n) = f(d)f(n/d). \tag{5}$$

The Dirichlet convolution of two arithmetical functions f_1 and f_2 is defined by $(f_1 * f_2)(n) = \sum_{d|n} f_1(d)f_2(n/d)$. The identity under the Dirichlet convolution is the arithmetical function δ defined as $\delta(1) = 1$ and $\delta(n) = 0$ for $n \geq 2$. An arithmetical function f is said to be a totient if $f = g * h^{-1}$, where g and h are completely multiplicative functions with $h(1) \neq 0$ and h^{-1} is the Dirichlet inverse of h.

Totients are rational arithmetical functions of order (1, 1) [18]. Euler's function ϕ is an example of a totient. For general accounts on arithmetical functions, see [4,40,45,46].

In this article we consider integer-valued arithmetical functions, i.e. functions $f: \mathbb{Z}^+ \to \mathbb{Z}$. Then, on the basis of (5), each multiplicative function f is order preserving in the sense that

$$d \| n \Rightarrow f(d) \mid f(n). \tag{6}$$

Further, it is known [14] that if f is a totient, then

$$d \mid n \Rightarrow f(d) \mid f(n). \tag{7}$$

As a special case of (7) we obtain the divisibility property (1) of Euler's totient function. In the next theorem we characterize all multiplicative functions satisfying (7).

THEOREM 3.1 *Let f be a multiplicative function. Then* (7) *holds if and only if for each prime p there exists a sequence of integers $\{c_i\}_{i=2}^{\infty}$ (depending on p) such that for all integers $e \geq 2$ we have*

$$f(p^e) = c_2 c_3 \cdots c_e f(p). \tag{8}$$

Proof Assume that (7) holds. Then, inductively,

$$\begin{aligned} f(p^2) &= c_2 f(p) \\ f(p^3) &= c_3 f(p^2) = c_3 c_2 f(p) \\ f(p^4) &= c_4 f(p^3) = c_4 c_3 c_2 f(p) \\ &\cdots \end{aligned}$$

we obtain (8). Assume then that (8) holds. We show that

$$f(p^a) \mid f(p^b) \tag{9}$$

for all primes p and integers $0 \leq a \leq b$. If $a = b$, then (9) holds. Assume that $0 \leq a < b$. Then (8) shows that

$$f(p^b) = c_2 c_3 \cdots c_b f(p) = c_{a+1} c_3 \cdots c_b f(p^a)$$

and consequently, (9) holds. Since f is multiplicative, (7) holds. ■

Example 3.1

(i) If f is completely multiplicative, then $f(p^e) = f(p)^e$ for all primes p and integers $e \geq 2$ [4]. Thus (8) holds with $c_i = f(p)$ for all $i \geq 2$.
(ii) If f is a totient, then there is an integer $a(p)$ such that $f(p^e) = f(p)a(p)^{e-1}$ for all primes p and integers $e \geq 2$ [14]. Thus (8) holds with $c_i = a(p)$ for all $i \geq 2$.

As noted in Section 1, $(\mathbb{Z}^+, \mid)$ is a poset and $(\mathbb{Z}, \mid)$ is a preposet. Further, it is easy to see that $(\mathbb{Z}^+, \|)$ is a poset and even a meet semilattice, see [18]. Thus $(\mathbb{Z}^+, \mid)$, $(\mathbb{Z}, \mid)$ and $(\mathbb{Z}^+, \|)$ induce topologies, say $\mathcal{Z}^+$, $\mathcal{Z}$ and $\mathcal{Z}_u^+$, on $\mathbb{Z}^+$, $\mathbb{Z}$ and $\mathbb{Z}^+$, respectively. Since $(\mathbb{Z}^+, \mid)$ and $(\mathbb{Z}^+, \|)$ are posets, $\mathcal{Z}^+$ and $\mathcal{Z}_u^+$ are T_0-topologies. Further, $(\mathbb{Z}^+, \mid)$

and ($\mathbb{Z}^+$, $\|$) are posets with least element 1 and ($\mathbb{Z}$, $|$) is a preposet with least elements ± 1; hence $\mathcal{Z}^+$, $\mathcal{Z}_u^+$ and $\mathcal{Z}$ are compact, connected topological spaces. Prime numbers are the atoms in ($\mathbb{Z}^+$, $|$), and prime powers are the atoms in ($\mathbb{Z}^+$, $\|$). Let $\mathbb{P}$ denote the set of all prime numbers, and let $\mathbb{P}_u = \{p^a : p \in \mathbb{P},\ a \in \mathbb{Z}^+\}$ denote the set of all prime powers. Then ($\mathbb{P}$, $|$) and ($\mathbb{P}_u$, $\|$) are antichains and thus induce T_1-topologies.

From Theorem 2.1 and Equation (6) we obtain the following theorem.

THEOREM 3.2 *Each multiplicative function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is continuous with respect to the topologies $\mathcal{Z}_u^+$ and $\mathcal{Z}$.*

From Theorem 2.1 and Equation (7) we obtain the following theorem.

THEOREM 3.3 *Each totient function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$.*

From Theorem 3.1 we obtain the following theorem.

THEOREM 3.4 *A multiplicative function f: $\mathbb{Z}^+ \to \mathbb{Z}$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$ if and only if for each prime p there exists a sequence of integers $\{c_i\}_{i=2}^{\infty}$ (depending on p) such that for all integers $e \geq 2$ equation* (8) *holds.*

Remark It is easy to see that $\mathcal{Z}^+ \subset \mathcal{Z}_u^+$. Thus, if $f : \mathbb{Z}^+ \to \mathbb{Z}$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$, then f is continuous with respect to the topologies $\mathcal{Z}_u^+$ and $\mathcal{Z}$. This means the trivial fact that if f satisfies the order preserving property (7), it satisfies the order preserving property (6).

4. Power GCD and LCM matrices

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers, and let a be a positive integer. The ath power GCD matrix (S^a) on S is defined as the $n \times n$ matrix whose ij entry is the ath power of the greatest common divisor of x_i and x_j. The ath power LCM matrix $[S^a]$ on S is defined analogously. A large number of papers have recently been published on these matrices, see [3,12,15–17, 22,27,30,31,33,43,50,51,53,55,56].

The power GCD and LCM matrices are symmetric and thus (S^a) divides (S^b) from the left if and only if (S^a) divides (S^b) from the right. Therefore, in this case, we may say that (S^a) divides (S^b), written as $(S^a) \mid (S^b)$. The same holds for the power LCM matrices $[S^a]$. A large number of papers have recently been published on the divisibility of GCD- and LCM-type matrices, see [7,9,12,19,21,24–26,28,30,32,37, 50–53,55,56]. Note that infinite divisibility of matrices studied in [6,29] is a concept different from divisibility of matrices.

Divisibility from the left (right) is a preorder on $M_n(\mathbb{Z})$. Thus it induces a topology, say $\mathcal{M}_n$, on $M_n(\mathbb{Z})$. The preposet $M_n(\mathbb{Z})$ under the left (right) divisibility possesses least elements. Therefore, $\mathcal{M}_n$ is a compact, connected topological space (note that least elements in the preposet $M_n(\mathbb{Z})$ are the invertible matrices in $M_n(\mathbb{Z})$. A matrix $U \in M_n(\mathbb{Z})$ is invertible in $M_n(\mathbb{Z})$ if and only if $\det(U) = \pm 1$. Such matrices U are referred to as unimodular matrices). The preposet $M_n(\mathbb{Z})$ under the left (right) divisibility is not a poset. Therefore, $\mathcal{M}_n$ is not a T_0-topology. This kind of topological approach has not hitherto been presented in the field of divisibility of GCD- and LCM-type matrices.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers. Then S is said to be a divisor chain [30] if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $x_{\sigma(1)} \mid x_{\sigma(2)} \mid \cdots \mid x_{\sigma(n)}$, and S is said consist of two coprime divisor chains [51] if there exists a partition $S = S_1 \cup S_2$ such that S_1 and S_2 are divisor chains and each element of S_1 is coprime to each element of S_2.

It is known [30,51] that if S is a divisor chain or consists of two coprime divisor chains with $1 \in S$, then

$$a \mid b \Rightarrow (S^a) \mid (S^b), \tag{10}$$

$$a \mid b \Rightarrow [S^a] \mid [S^b]. \tag{11}$$

Note that (10) and (11) imply that

$$a \mid b \Rightarrow \det(S^a) \mid \det(S^b), \tag{12}$$

$$a \mid b \Rightarrow \det[S^a] \mid \det[S^b]. \tag{13}$$

From Theorem 2.1 and Equations (10) and (11) we obtain the following theorem.

THEOREM 4.1 *Suppose that S is a divisor chain or consists of two coprime divisor chains with $1 \in S$.*

(i) *The function $f: \mathbb{Z}^+ \to M_n(\mathbb{Z})$, $f(a) = (S^a)$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{M}_n$.*
(ii) *The function $g: \mathbb{Z}^+ \to M_n(\mathbb{Z})$, $g(a) = [S^a]$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{M}_n$.*

Tan and Lin [52] say that the set S consists of finitely many quasi-coprime divisor chains if one can partition S as $S = S_1 \cup S_2 \cup \cdots \cup S_k$, where $k \geq 1$ is an integer and all S_i $(1 \leq i \leq k)$ are divisor chains such that $\gcd(\max(S_i), \max(S_j)) = \gcd(S)$ for any $1 \leq i \neq j \leq k$. If S satisfies the assumption of Theorem 4.1, that is, if S is a divisor chain or consists of two coprime divisor chains with $1 \in S$, then S consists of finitely many quasi-coprime divisor chains. Tan and Lin [52] show that if S consists of finitely many quasi-coprime divisor chains with $\gcd(S) \in S$, then Equations (12) and (13) hold. Applying Theorem 2.1 we thus obtain the following theorem.

THEOREM 4.2 *Suppose that S consists of finitely many quasi-coprime divisor chains with* $\gcd(S) \in S$.

(i) *The function $f: \mathbb{Z}^+ \to \mathbb{Z}$, $f(a) = \det(S^a)$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$.*
(ii) *The function $g: \mathbb{Z}^+ \to \mathbb{Z}$, $g(a) = \det[S^a]$ is continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$.*

5. Smith normal form

A matrix $A \in M_n(\mathbb{Z})$ is said to be equivalent to $B \in M_n(\mathbb{Z})$ if $A = UBV$ for some unimodular $U, V \in M_n(\mathbb{Z})$. This relation is reflexive, symmetric and transitive. Every matrix $A \in M_n(\mathbb{Z})$ is equivalent to a diagonal matrix

$$D = \operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0),$$

where $r = \text{rank}(A)$, $d_i \in \mathbb{Z}^+$, $1 \le i \le r$, and $d_i | d_{i+1}$, $1 \le i \le r-1$. This diagonal matrix is unique and it is called the Smith normal form of A. The research on the area of Smith normal form and its applications started when Smith [47] developed a systematic process for finding general solution for certain systems of linear equations (or congruences). Since then lots of results concerning the Smith normal form have been published [34,41]. We next consider the Smith normal form of GCD matrices.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers, and let f be an integer-valued arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having $f((x_i, x_j))$ as its ij entry, where $f((x_i, x_j))$ is f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j. Let $E = (e_{ij})$ and $D = \text{diag}(d_1, d_2, \ldots, d_n)$ denote the $n \times n$ matrices defined as

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \,|x_i, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_i = (f * \mu)(x_i)$, where μ is the Möbius function. If the set S is factor closed (i.e. for each $x_i \in S$, each positive factor of x_i is also in S), then the matrix $(S)_f$ can be written as

$$(S)_f = EDE^T, \tag{14}$$

see [35]. This shows that $(S)_f$ is equivalent to

$$D = \text{diag}((f * \mu)(x_1), (f * \mu)(x_2), \ldots, (f * \mu)(x_n)). \tag{15}$$

Applying Theorem 2.1, we now obtain the following theorem.

THEOREM 5.1 *Suppose that* $S = \{1, p, p^2, \ldots, p^e\}$, *where* p *is a prime number and* $e \in \mathbb{Z}^+$. *If* $f * \mu$ *is continuous with respect to the topologies* $\mathcal{Z}^+$ *and* $\mathcal{Z}$, *then the Smith normal form of* $(S)_f$ *is*

$$D = \text{diag}((f * \mu)(1), (f * \mu)(p), (f * \mu)(p^2), \ldots, (f * \mu)(p^e)). \tag{16}$$

Example 5.1 If f is completely multiplicative, then the function $f * \mu$ in Theorem 5.1 is a totient and thus continuous with respect to the topologies $\mathcal{Z}^+$ and $\mathcal{Z}$.

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